Topological gravity and transgression holography

Patricio Salgado[1*], **Richard J. Szabo**[2†] and **Omar Valdivia**[1,2‡]

[1]*Departamento de Física*
Universidad de Concepción
Casilla 160-C, Concepción, Chile

[2]*Department of Mathematics*
Heriot–Watt University
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K.
and *Maxwell Institute for Mathematical Sciences, Edinburgh, U.K.*
and *The Tait Institute, Edinburgh, U.K.*

Abstract

We show that Poincaré-invariant topological gravity in even dimensions can be formulated as a transgression field theory in one higher dimension whose gauge connections are associated to linear and nonlinear realizations of the Poincaré group $ISO(d-1,1)$. The resulting theory is a gauged WZW model whereby the transition functions relating gauge fields live in the coset $\frac{ISO(d-1,1)}{SO(d-1,1)}$. The coordinate parametrizing the coset space is identified with the scalar field in the adjoint representation of the gauge group of the even-dimensional topological gravity theory. The supersymmetric extension leads to topological supergravity in two dimensions starting from a transgression field theory which is invariant under the supersymmetric extension of the Poincaré group in three dimensions. We also apply this construction to a three-dimensional Chern–Simons theory of gravity which is invariant under the Maxwell algebra and obtain the corresponding WZW model.

[*]Email: `pasalgad@udec.cl`
[†]Email: `R.J.Szabo@hw.ac.uk`
[‡]Email: `ov32@hw.ac.uk`

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1   Introduction and summary

Gauge interactions, such as those which govern the Standard Model of particle physics, are commonly based on a dynamical structure which naturally encodes the property that nature should be invariant under a group of transformations acting on each point of spacetime, i.e., a local gauge symmetry. Although it is not guaranteed that gauge invariant field theories are renormalizable, the only renormalizable models describing nature are gauge theories, and thus the gauge symmetry principle seems to be a key ingredient in physically testable theories. The gravitational interaction, in contrast, has stubbornly resisted quantization. General relativity seems to be the consistent framework compatible with the idea that physics should be insensitive to the choice of coordinates or the state of motion of any observer; this is expressed mathematically as invariance under reparametrizations or local diffeomorphisms. Although this invariance constitutes a local symmetry, it does not qualify as a gauge symmetry. The reason is that gauge transformations act on the fields while diffeomorphisms act on their arguments, i.e., on the coordinates. A systematic way to circumvent this obstruction is by using the tangent space representation; in this framework gauge transformations constitute changes of frames which leave the coordinates unchanged. However, general relativity is not invariant under local translations, except by a special accident in three spacetime dimensions where the Einstein–Hilbert action is purely topological.

In refs. [1, 2, 3] the classification of topological gauge theories for gravity is presented. The natural gauge groups G considered are the anti-de Sitter group $SO(d-1,2)$, the de Sitter group $SO(d,1)$, and the Poincaré group $ISO(d-1,1)$ in d spacetime dimensions depending on the sign of the cosmological constant: $-1,+1,0$ respectively. In odd dimensions $d = 2n+1$, the gravitational theories are constructed in terms of secondary characteristic classes called Chern–Simons forms. Chern–Simons forms are useful objects because they lead to gauge invariant theories (modulo boundary terms). They also have a rich mathematical structure similar to those of the (primary) characteristic classes that arise in Yang–Mills theories: they are constructed in terms of a gauge potential which descends from a connection on a principal G-bundle. In even dimensions, there is no natural candidate such as the Chern–Simons forms; hence in order to construct an invariant $2n$-form, the product of n field strengths is not sufficient and requires the insertion of a scalar multiplet ϕ^a transforming in the adjoint representation of the gauge group G. This requirement ensures gauge invariance but it threatens the topological origin of the theory.

In this paper we show that even-dimensional topological gravity can be formulated as a transgression field theory [4, 5] which is invariant under the Poincaré group. The gauge connections are considered as valued in both the Lie algebras associated to linear and nonlinear realizations of the gauge group. The resulting theory is a gauged Wess–Zumino–Witten (WZW) model [6, 7] where the scalar field ϕ is now identified with the coset parameter of the nonlinear realization of the Poincaré group $ISO(d-1,1)$. This identification allows the construction of topological gravity as a holographic dual of a transgression gauge field theory in odd dimensions. However, the transformation laws for the coset field break translation invariance and therefore the residual symmetry is constrained to the Lorentz subgroup $SO(d-1,1)$. This is not a huge obstruction in the sense that one can restore the full Poincaré symmetry by considering the coset fields as transforming in the adjoint representation of the gauge group. By similar arguments we also compute the transgression action for the $\mathcal{N} = 1$ Poincaré supergroup in three dimensions, and

show that the resulting action is the one proposed in [1]; it would be interesting to work out the generalizations of this supergravity theory to higher dimensions. Finally, we apply this construction to obtain a gauged WZW model associated to the Maxwell algebra in two dimensions; the resulting theory generalizes the topological gravity theory proposed by [1] and we show that, in order to obtain the requisite invariant tensor associated to the Maxwell algebra, an S-expansion of the AdS algebra in three dimensions in required. There are various interesting extensions of our work that can be considered. For example, it would be interesting to construct a suitable transgression field theory in three dimensions for the Liouville theories considered in e.g. [8] which are invariant under the BMS algebra and which are dual to asymptotically flat Chern–Simons gravity. Moreover, it would interesting to construct the transgression field theories based on the AdS group $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and more generally on gauge groups based on non-compact real forms of $SL(n,\mathbb{C}) \times SL(n,\mathbb{C})$ in three dimensions which describe the couplings of higher spin n fields to gravity [9], and to study the dual WZW models; in such instances, however, our approach based on nonlinear realisations of the gauge group does not appear to be very tractable.

The structure of this paper is as follows. In Section 2 we review some aspects of topological gravity and its relation with Lanczos–Lovelock theories of gravity. In Section 3 we review the formalism of nonlinear realizations of Lie groups and its application to gravitational theories. Section 4 introduces transgression forms and presents the main results of this investigation. As a representative example of how to incorporate fermions into our construction, in Section 5 we derive the supersymmetric extension of the topological gravity action in the two-dimensional case. Section 6 contains a brief application in which we construct the gauged WZW model associated to the Maxwell algebra. Three appendices at the end of the paper contain some technical details about the construction of Chern–Simons gravity actions, our conventions for spinors, and the S-expansion method for Lie algebras.

2 Topological gravity and Lanczos–Lovelock theory

2.1 Topological gauge theories of gravity

Topological gauge theories of gravity were classified in refs. [1, 2, 3]. The natural gauge groups G involved in the classification are given by

$$
G \quad : \quad
\begin{array}{|l|c|c|}
\hline
\text{AdS} & SO(d-1,2) & \Lambda < 0 \\
\hline
\text{dS} & SO(d,1) & \Lambda > 0 \\
\hline
\text{Poincaré} & ISO(d-1,1) & \Lambda = 0 \\
\hline
\end{array}
\qquad (2.1)
$$

depending on the spacetime dimension d and the sign of the cosmological constant Λ. These gauge groups are the smallest nontrivial choices which contain the Lorentz symmetry $SO(d-1,1)$ as well as symmetries analogous to local translations.

Throughout this paper, we shall let $\mathcal{P}$ denote a principal G-bundle over a smooth manifold $\mathcal{M}$ of dimension d. Let $\mathcal{A} \in \Omega^1(U, \mathfrak{g})$ be a local gauge potential with values in the Lie algebra $\mathfrak{g}$ of G, obtained as the pull-back by a local section $\sigma : U \to \mathcal{P}$, $U \subset \mathcal{M}$ of a one-form connection $\theta \in \Omega^1(\mathcal{P}, \mathfrak{g})$ as

$$
\mathcal{A} = \sigma^* \theta \ . \qquad (2.2)
$$

In odd dimensions $d = 2n + 1$, the action for topological gravity is written in terms of a Chern–Simons form defined by

$$S^{(2n+1)}[\mathcal{A}] = \kappa \int_{\mathcal{M}} L_{CS}^{(2n+1)}(\mathcal{A}) = \kappa\,(n+1) \int_{\mathcal{M}} \int_0^1 dt \, \left\langle \mathcal{A} \wedge \left(t\,d\mathcal{A} + t^2\,\mathcal{A} \wedge \mathcal{A}\right)^n \right\rangle \,. \quad (2.3)$$

Here $\kappa \in \mathbb{R}$ is a constant and $\langle - \rangle : \mathfrak{g}^{\otimes(n+1)} \to \mathbb{R}$ is a G-invariant symmetric polynomial of rank $n + 1$ which is determined once an explicit presentation for $\mathfrak{g}$ is chosen. Note that the Chern–Simons form $L_{CS}^{(2n+1)}(\mathcal{A})$ is not globally-defined (unless $\mathcal{P}$ is trivial) and the gauge theory specified by eq. (2.3) is regarded as defined by a sheaf of local Lagrangians.

In even dimensions there is no topological candidate such as the Chern–Simons form. In fact, the exterior product of n field strengths makes the required $2n$-form in a $2n$-dimensional spacetime, but in order to obtain a gauge invariant differential $2n$-form, a scalar multiplet ϕ^a with $a = 1, \ldots, 2n + 1$ transforming in the adjoint representation of the gauge group must be added and the action is given by

$$S^{(2n)}[\mathcal{A}, \phi] = \kappa \int_{\mathcal{M}} \left\langle \mathcal{F}^n \wedge \phi \right\rangle \,. \quad (2.4)$$

Here $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the curvature two-form associated to the gauge potential $\mathcal{A}$. Note that here the Lagrangian $\left\langle \mathcal{F}^n \wedge \phi \right\rangle$ is a global differential form on $\mathcal{M}$. This topological action has interesting applications; for instance, in two dimensions it describes the Liouville theory of gravity from a local Lagrangian [10, 11].

2.2 Lanczos–Lovelock gravity

The most general Lagrangian in d dimensions which is compatible with the Einstein–Hilbert action for gravity is a polynomial of degree $[d/2]$ in the curvatures known as the Lanczos–Lovelock Lagrangian [12, 13, 14, 15, 16]. Lanczos–Lovelock theories share the same fields, symmetries and local degrees of freedom of General Relativity. The Lagrangian is built from the vielbein e^a and the spin connection ω^{ab} via the Riemann curvature two-form $R^{ab} = d\omega^{ab} + \omega^a_{\ c} \wedge \omega^{cb}$, leading to the action

$$S_{LL}^{(d)} = \int_{\mathcal{M}} \sum_{p=0}^{[d/2]} \alpha_p \, \epsilon_{a_1 \cdots a_d} \, R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \cdots \wedge e^{a_d} \,. \quad (2.5)$$

Here α_p are arbitrary parameters that cannot be fixed from first principles. However, in ref. [17] it is shown that by requiring the equations of motion to uniquely determine the dynamics for as many components of the independent fields as possible, one can fix α_p (in any dimension) in terms of the gravitational and cosmological constants.

In $d = 2n$ dimensions the parameters α_p are given by

$$\alpha_p = \alpha_0 \, (2\gamma)^p \, \binom{n}{p} \quad (2.6)$$

and the Lagrangian takes a Born–Infeld form. The Lanczos–Lovelock action constructed in this dimension is only invariant under the Lorentz symmetry $SO(2n - 1, 1)$. In odd dimensions $d = 2n + 1$ the coefficients are given by

$$\alpha_p = \alpha_0 \, \frac{(2n - 1)\,(2\gamma)^p}{2n - 2p - 1} \, \binom{n-1}{p} \,. \quad (2.7)$$

Here

$$\alpha_0 = \frac{\kappa}{d\,l^{d-1}} \qquad \text{and} \qquad \gamma = -\text{sgn}(\Lambda)\,\frac{l^2}{2} \tag{2.8}$$

with κ an arbitrary dimensionless constant, and l is a length parameter related to the cosmological constant by

$$\Lambda = \pm \frac{(d-1)\,(d-2)}{2l^2} \ . \tag{2.9}$$

With this choice of coefficients, the Lanczos–Lovelock Lagrangian for $d = 2n + 1$ coincides exactly with a Chern–Simons form for the AdS group $SO(2n, 2)$. This means that the exterior derivative of the Lanczos–Lovelock Lagrangian corresponds to a $2n+2$-dimensional Euler density. This is the reason why there is no analogous construction in even dimensions: There are no known topological invariants in odd dimensions which can be constructed in terms of exterior products of curvatures alone. However, in ref. [18] a Lanczos–Lovelock theory genuinely invariant under the AdS group, in any dimension, is proposed. The construction is based on the Stelle–West mechanism [19, 20], which is an application of the theory of nonlinear realizations of Lie groups to gravity.

3   Nonlinear realizations of Lie groups

3.1   Nonlinear gauge theories

Nonlinear realizations of Lie groups were introduced in refs. [21, 22]. Following these references, let G be a (super-)Lie group of transformations of dimension n and $\mathfrak{g}$ its Lie algebra. Let H be a stability subgroup of G of dimension $n - d$ whose Lie algebra $\mathfrak{h}$ is generated by $\{\mathsf{V}_i\}_{i=1}^{n-d}$. Let us denote by $\mathfrak{p}$ the vector subspace generated by the remaining generators of $\mathfrak{g}$, denoted $\{\mathsf{P}_l\}_{l=1}^{d}$, such that there is a vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Since $\mathfrak{h}$ is a subalgebra, one has $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. We will further assume that $\mathfrak{p}$ can be chosen in such a way that it defines a representation of H, so that $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. With this decomposition, any element $g_0 \in G$ can always be uniquely written as

$$g_0 = \mathrm{e}^{\,\zeta\cdot\mathsf{P}}\,h \tag{3.1}$$

where $h \in H$ and $\mathrm{e}^{\,\zeta\cdot\mathsf{P}} = \mathrm{e}^{\,\zeta^l\,\mathsf{P}_l} \in G/H$ with $l = 1, \ldots, d$. The local coordinates ζ parametrize the coset space G/H. By virtue of eq. (3.1), the action of g_0 on the coset space G/H is given by

$$g_0\,\mathrm{e}^{\,\zeta\cdot\mathsf{P}} = \mathrm{e}^{\,\zeta'\cdot\mathsf{P}}\,h_1 \ . \tag{3.2}$$

This expression allow us to obtain ζ' and h_1 as certain nonlinear functions of g_0 and ζ,

$$\zeta' = \zeta'(g_0, \zeta) \qquad \text{and} \qquad h_1 = h_1(g_0, \zeta) \ . \tag{3.3}$$

For g_0 close to the identity, eq. (3.2) reads

$$\mathrm{e}^{\,-\zeta\cdot\mathsf{P}}\,(g_0 - 1)\,\mathrm{e}^{\,\zeta\cdot\mathsf{P}} - \mathrm{e}^{\,-\zeta\cdot\mathsf{P}}\,\delta\,\mathrm{e}^{\,\zeta\cdot\mathsf{P}} = h_1 - 1\ , \tag{3.4}$$

allowing us to obtain the variation $\delta\zeta = \zeta' - \zeta$ under the infinitesimal action of G. Note that if we restrict G to the subgroup H, the nonlinear representation becomes linear: If $g_0 = h_0 \in H$, then eq. (3.2) takes the form

$$\mathrm{e}^{\,\zeta'\cdot\mathsf{P}}\,h_1 = h_0\,\mathrm{e}^{\,\zeta\cdot\mathsf{P}} = \left(h_0\,\mathrm{e}^{\,\zeta\cdot\mathsf{P}}\,h_0^{-1}\right)h_0 \tag{3.5}$$

and since $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$ the term $h_0 \, \mathrm{e}^{\zeta \cdot \mathsf{P}} \, h_0^{-1}$ is an exponential in the generators of $\mathfrak{p}$, implying

$$h_1 = h_0 \, , \tag{3.6}$$

$$\mathrm{e}^{\zeta' \cdot \mathsf{P}} = h_0 \, \mathrm{e}^{\zeta \cdot \mathsf{P}} \, h_0^{-1} \, , \tag{3.7}$$

where the transformation from ζ to ζ' in eq. (3.7) is linear. On the other hand, if

$$g_0 = \mathrm{e}^{\zeta_0 \cdot \mathsf{P}} \ \in \ G/H \, , \tag{3.8}$$

then eq. (3.2) becomes

$$\mathrm{e}^{\zeta_0 \cdot \mathsf{P}} \, \mathrm{e}^{\zeta \cdot \mathsf{P}} = \mathrm{e}^{\zeta' \cdot \mathsf{P}} \, h_1 \tag{3.9}$$

which is a nonlinear transformation in the coset coordinate ζ.

The construction of Lagrangians which are invariant under local gauge transformations usually involves the introduction of a set of gauge fields associated with the generators of $\mathfrak{g}$. Here, as in the case of linear representations, a nonlinear gauge potential $\bar{\mathcal{A}}$ must be introduced in order to guarantee that the derivatives of the fields $\zeta, \bar{\varphi}$ transform covariantly, where $\bar{\varphi}$ are coordinates on the subgroup H. The linear gauge potential $\mathcal{A}$ can be naturally decomposed into gauge fields associated to H and G/H as

$$\mathcal{A} = v^i \, \mathsf{V}_i + p^l \, \mathsf{P}_l \, . \tag{3.10}$$

Introducing the nonlinear gauge potential $\bar{\mathcal{A}}$, we can now write the nonlinear gauge fields

$$\bar{\mathcal{A}} = \bar{v}^i \, \mathsf{V}_i + \bar{p}^l \, \mathsf{P}_l \, . \tag{3.11}$$

The linear and nonlinear gauge potentials are related by [21, 22]

$$\bar{v}^i \, \mathsf{V}_i + \bar{p}^l \, \mathsf{P}_l = \mathrm{e}^{\zeta \cdot \mathsf{P}} \left(\mathrm{d} + v^i \, \mathsf{V}_i + p^l \, \mathsf{P}_l \right) \mathrm{e}^{-\zeta \cdot \mathsf{P}} \, . \tag{3.12}$$

This relation has exactly the form of a gauge transformation by $\mathrm{e}^{-\zeta \cdot \mathsf{P}} \in G/H$. The transformation relations for $\bar{v} = \bar{v}(\zeta, \mathrm{d}\zeta)$, $\bar{p} = \bar{p}(\zeta, \mathrm{d}\zeta)$ are obtained using eqs. (3.1, 3.2) and are given by

$$\bar{v}' = h_1^{-1} \, \bar{v} \, h_1 \, , \tag{3.13}$$

$$\bar{p}' = h_1^{-1} \, \bar{p} \, h_1 + h_1^{-1} \, \mathrm{d} h_1 \tag{3.14}$$

with $h_1(\zeta, \zeta_0) \in H$. Eqs. (3.13, 3.14) show that the nonlinear fields $\bar{v} = \bar{v}(\zeta, \mathrm{d}\zeta)$ and $\bar{p} = \bar{p}(\zeta, \mathrm{d}\zeta)$ transform as a tensor and as a connection respectively under the action of $h_1(\zeta, \zeta_0) \in H$. Since h_1 depends on ζ, any H-invariant expression written in terms of v and p will be automatically invariant under the full group G, provided one replaces the linear gauge fields v and p by their nonlinear versions $\bar{v}$ and $\bar{p}$. We now make use of the nonlinear gauge fields and their properties to define the covariant derivative respect to the group G as

$$D_{\bar{p}} := \mathrm{d} + \bar{p} \, , \tag{3.15}$$

and the corresponding curvature two-form whose components are given by

$$T = D_{\bar{p}} \bar{v} \qquad \text{and} \qquad R = \mathrm{d}\bar{p} + \bar{p} \wedge \bar{p} \, . \tag{3.16}$$

7

3.2 SWGN formalism

The Stelle–West–Grignani–Nardelli (SWGN) formalism [19, 20] is an application of the theory of nonlinear realizations of Lie groups to gravity. In particular, it allows the construction of the Lanczos–Lovelock theory of gravity which is genuinely invariant under the anti-de Sitter group $G = SO(d-1, 2)$. This model is discussed in ref. [23] and it is described by the action

$$\mathrm{S}_{\mathrm{SW}}^{(d)} = \int_{\mathcal{M}} \sum_{p=0}^{[d/2]} \alpha_p \, \epsilon_{a_1 \cdots a_d} \, \bar{R}^{a_1 a_2} \wedge \cdots \wedge \bar{R}^{a_{2p-1} a_{2p}} \wedge \bar{e}^{a_{2p+1}} \wedge \cdots \wedge \bar{e}^{a_d} \,. \tag{3.17}$$

Here $\bar{R}^{ab}$ and $\bar{e}^a$ are nonlinear gauge fields and the coefficients α_p are given by either eq. (2.6) or eq. (2.7) depending on the dimension of the spacetime. Using eq. (3.12) we get

$$\bar{e}^a = \Omega^a{}_b (\cosh z) \, e^b + \Omega^a{}_b \left(\frac{\sinh z}{z} \right) D_\omega \phi^b \,, \tag{3.18}$$

$$\bar{R}^{ab} = \mathrm{d}\bar{\omega}^{ab} + \bar{\omega}^a{}_c \wedge \bar{\omega}^{cb} \,, \tag{3.19}$$

$$\bar{\omega}^{ab} = \omega^{ab} + \frac{\sigma}{l^2} \left(\frac{\sinh z}{z} \left(\phi^a e^b - \phi^b e^a \right) + \frac{\cosh z - 1}{z^2} \left(\phi^a D_\omega \phi^b - \phi^b D_\omega \phi^a \right) \right) \,, \tag{3.20}$$

where e^a and ω^{ab} are the usual vielbein and spin connection, respectively. We have defined

$$D_\omega \phi^a := \mathrm{d}\phi^a + \omega^a{}_b \, \phi^b \,,$$

$$z := \frac{\phi}{l} = \frac{\sqrt{\phi^a \phi_a}}{l} \,,$$

$$\Omega^a{}_b(u) := u \, \delta^a{}_b + (1-u) \, \frac{\phi^a \phi_b}{\phi^2} \,, \tag{3.21}$$

where l is the radius of curvature of AdS and ϕ^a are the AdS coordinates which parametrize the coset space $\frac{SO(d-1,2)}{SO(d-1,1)}$. In this scheme, this coordinate carries no dynamics as any value that we pick for it is equivalent to a gauge fixing condition which breaks the symmetry from AdS to the Lorentz subgroup. This is best seen using the equations of motion; they are the same as those for the ordinary Lanczos–Lovelock theory where the vielbein e^a and the spin connection ω^{ab} are replaced by their nonlinear versions $\bar{e}^a$ and $\bar{\omega}^{ab}$ given in eqs. (3.18, 3.20).

In odd dimensions $d = 2n+1$, the Chern–Simons action written in terms of the linear gauge fields e^a and ω^{ab} with values in the Lie algebra of $SO(2n, 2)$ differs only by a boundary term from that written using the nonlinear gauge fields $\bar{e}^a$ and $\bar{\omega}^{ab}$. This is by virtue of eq. (3.12) which has the form of a gauge transformation

$$\mathcal{A} \longmapsto \bar{\mathcal{A}} = g^{-1} \, (\mathrm{d} + \mathcal{A}) \, g \tag{3.22}$$

with $g = \mathrm{e}^{-\phi^a \, \mathsf{P}_a} \in \frac{SO(2n,2)}{SO(2n,1)}$. Alternatively, since $\bar{\mathcal{F}} = g^{-1} \mathcal{F} g$ we have

$$\mathrm{d}\, \mathrm{L}_{\mathrm{CS}}^{(2n+1)} \left(\bar{\mathcal{A}} \right) = \left\langle \bar{\mathcal{F}}^{n+1} \right\rangle = \left\langle \mathcal{F}^{n+1} \right\rangle = \mathrm{d}\, \mathrm{L}_{\mathrm{CS}}^{(2n+1)} \left(\mathcal{A} \right) \tag{3.23}$$

and hence both Lagrangians may locally differ only by a total derivative.

3.3 Poincaré gravity

In odd spacetime dimensions $d = 2n + 1$, Poincaré gravity is a Chern–Simons theory for the gauge group $G = ISO(2n, 1)$. This group can be obtained by performing an Inönü–Wigner contraction of the AdS group in odd dimensions $SO(2n, 2)$. The bulk part of the Lagrangian can still be recovered in the limit $l \to \infty$ from the Lanczos–Lovelock series in the case $d = 2n+1$ and $p = n$. However, there is an extra boundary term which arises once the computation of the relevant Chern–Simons form is carried out; see Appendix A for details and conventions. The resulting Lagrangian is then given by

$$
\begin{aligned}
\mathrm{L}_{\mathrm{CS}}^{(2n+1)}\left(\mathcal{A}\right) = {} & \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2n-1} a_{2n}} \wedge e^{a_{2n+1}} \\
& - n \, \mathrm{d} \int_0^1 \mathrm{d}t \, t^n \, \epsilon_{a_1 \cdots a_{2n+1}} R_t^{a_1 a_2} \wedge \cdots \wedge R_t^{a_{2n-3} a_{2n-2}} \wedge \omega^{a_{2n-1} a_{2n}} \wedge e^{a_{2n+1}}
\end{aligned} \tag{3.24}
$$

where $R_t^{ab} = \mathrm{d}\omega^{ab} + t \, \omega^a{}_c \wedge \omega^{cb}$. Under infinitesimal local gauge transformations with parameter $\lambda = \frac{1}{2} \kappa^{ab} \mathsf{J}_{ab} + \rho^a \mathsf{P}_a$, the gauge fields transform as

$$
\delta e^a = -D_\omega \rho^a + \kappa^a{}_b \, e^b \qquad \text{and} \qquad \delta \omega^{ab} = -D_\omega \kappa^{ab} \,, \tag{3.25}
$$

and these transformations leave eq. (3.24) invariant modulo a total derivative.

The nonlinear Lagrangian can be obtained using eqs. (3.18, 3.20) in the limit $l \to \infty$ and substituting into eq. (3.24) to obtain

$$
\begin{aligned}
\mathrm{L}_{\mathrm{CS}}^{(2n+1)}\left(\bar{\mathcal{A}}\right) = {} & \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2n-1} a_{2n}} \wedge \left(e^{a_{2n+1}} + D_\omega \phi^{a_{2n+1}}\right) \\
& - n \, \mathrm{d} \int_0^1 \mathrm{d}t \, t^n \, \epsilon_{a_1 \cdots a_{2n+1}} R_t^{a_1 a_2} \wedge \cdots \wedge R_t^{a_{2n-3} a_{2n-2}} \wedge \omega^{a_{2n-1} a_{2n}} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge \left(e^{a_{2n+1}} + D_\omega \phi^{a_{2n+1}}\right) \,.
\end{aligned} \tag{3.26}
$$

The gauge transformations for the coset field ϕ can be obtained from eq. (3.4) using $g_0 - 1 = -\phi^a \mathsf{P}_a$. In this case one shows that under local Poincaré translations the coset field ϕ transforms as $\delta\phi^a = \rho^a$. One can directly check, as in the case of the linear Lagrangian, that eq. (3.26) remains unchanged under gauge transformations up to a total derivative.

4 Topological gravity as a transgression field theory

4.1 Transgression forms as global Lagrangians

In this section we show that the topological action for gravity in $2n$ dimensions given in eq. (2.4) can be obtained from a $(2n + 1)$-dimensional transgression field theory which is genuinely invariant under the Poincaré group $G = ISO(2n, 1)$. Transgression forms are generalizations of Chern–Simons forms. They are gauge invariant objects and use, in addition to the gauge potential $\mathcal{A}$, a second Lie algebra valued gauge potential $\bar{\mathcal{A}}$. Due to their full invariance property they are good candidates for the construction of action principles by regarding $\mathcal{A}$ and $\bar{\mathcal{A}}$ as fundamental fields [24, 25, 26, 27].

Let $\mathcal{A}$ and $\bar{\mathcal{A}}$ be two $\mathfrak{g}$-valued connections. The transgression field theory is defined by the action

$$
\mathrm{S}_{\mathrm{T}}^{(2n+1)}\left[\mathcal{A}, \bar{\mathcal{A}}\right] = \kappa \int_{\mathcal{M}} Q_{\mathcal{A} \leftarrow \bar{\mathcal{A}}}^{(2n+1)} = \kappa \, (n+1) \int_{\mathcal{M}} \int_0^1 \mathrm{d}t \, \left\langle \left(\mathcal{A} - \bar{\mathcal{A}}\right) \wedge \mathcal{F}_t^n \right\rangle \tag{4.1}
$$

where $\mathcal{F}_t = \mathrm{d}\mathcal{A}_t + \mathcal{A}_t \wedge \mathcal{A}_t$ and $\mathcal{A}_t = \bar{\mathcal{A}} + t\left(\mathcal{A} - \bar{\mathcal{A}}\right)$ is a connection which interpolates between the two independent gauge potentials $\mathcal{A}$ and $\bar{\mathcal{A}}$. It is easy to check that the Chern–Simons form can be recovered in the special limit $\bar{\mathcal{A}} = 0$; in contrast, for dynamical gauge potentials the transgression form $Q^{(2n+1)}_{\mathcal{A} \leftarrow \bar{\mathcal{A}}}$ is a globally defined differential form on $\mathcal{M}$. Transgression forms satisfy two important properties:

- Triangle equation:
$$Q^{(2n+1)}_{\mathcal{A} \leftarrow \bar{\mathcal{A}}} = Q^{(2n+1)}_{\mathcal{A} \leftarrow \tilde{\mathcal{A}}} - Q^{(2n+1)}_{\bar{\mathcal{A}} \leftarrow \tilde{\mathcal{A}}} - \mathrm{d}Q^{(2n)}_{\mathcal{A} \leftarrow \bar{\mathcal{A}} \leftarrow \tilde{\mathcal{A}}} \, . \tag{4.2}$$

- Antisymmetry:
$$Q^{(2n+1)}_{\mathcal{A} \leftarrow \bar{\mathcal{A}}} = -Q^{(2n+1)}_{\bar{\mathcal{A}} \leftarrow \mathcal{A}} \, . \tag{4.3}$$

The first property splits a transgression form into the sum of two transgression forms depending on an intermediate connection $\tilde{\mathcal{A}}$ plus an exact form with [28]

$$Q^{(2n)}_{\mathcal{A} \leftarrow \bar{\mathcal{A}} \leftarrow \tilde{\mathcal{A}}} := n\,(n+1) \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s \, \left\langle \left(\mathcal{A} - \bar{\mathcal{A}}\right) \wedge \left(\bar{\mathcal{A}} - \tilde{\mathcal{A}}\right) \wedge \mathcal{F}_{st}^{n-1} \right\rangle \tag{4.4}$$

where $\mathcal{F}_{st} = \mathrm{d}\mathcal{A}_{st} + \mathcal{A}_{st} \wedge \mathcal{A}_{st}$ with $\mathcal{A}_{st} = \tilde{\mathcal{A}} + s\left(\mathcal{A} - \bar{\mathcal{A}}\right) + t\left(\bar{\mathcal{A}} - \tilde{\mathcal{A}}\right)$. Without any loss of generality, it is always possible to impose $\tilde{\mathcal{A}} = 0$ so that the transgression form becomes the difference of two Chern–Simons forms plus a boundary term

$$Q^{(2n+1)}_{\mathcal{A} \leftarrow \bar{\mathcal{A}}} = \mathrm{L}^{(2n+1)}_{\mathrm{CS}}\left(\mathcal{A}\right) - \mathrm{L}^{(2n+1)}_{\mathrm{CS}}\left(\bar{\mathcal{A}}\right) - \mathrm{d}\,\mathrm{B}^{(2n)}\left(\mathcal{A}, \bar{\mathcal{A}}\right) \tag{4.5}$$

where $\mathrm{B}^{(2n)}\left(\mathcal{A}, \bar{\mathcal{A}}\right) = Q^{(2n)}_{\mathcal{A} \leftarrow \bar{\mathcal{A}} \leftarrow 0}$. This equation illustrates that the global form $Q^{(2n+1)}_{\mathcal{A} \leftarrow \bar{\mathcal{A}}}$ can be interpreted as a globalization of the sheaf of local Lagrangians $\mathrm{L}^{(2n+1)}_{\mathrm{CS}}\left(\mathcal{A}\right) = Q^{(2n+1)}_{\mathcal{A} \leftarrow 0}$, with $\bar{\mathcal{A}}$ regarded as a background reference connection; in particular, on a local trivialization of $\mathcal{P}$ one can set $\bar{\mathcal{A}} = 0$.

4.2 Topological gravity actions

Let now $\mathcal{M}$ be a manifold of dimension $d = 2n+1$ with boundary $\partial\mathcal{M}$. Let $\mathcal{A}$ and $\bar{\mathcal{A}}$ be the linear and nonlinear one-form gauge potentials both taking values in the Lie algebra $\mathfrak{g} = \mathfrak{iso}(2n, 1)$. In the following we assume that both gauge potentials can be obtained as the pull-back by a local section σ of a one-form connection θ defined on a nontrivial principal G-bundle $\mathcal{P}$ over $\mathcal{M}$. This means that they are related by a gauge transformation which we take to be given by $g = \mathrm{e}^{-\phi^a\,\mathsf{P}_a} \in G/H$, where $H = SO(2n, 1)$ is the Lorentz subgroup.

From eq. (4.5) we see that the transgression action for a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ is given by

$$\mathrm{S}^{(2n+1)}_{\mathrm{T}}[\mathcal{A}, \bar{\mathcal{A}}] = \kappa \int_{\mathcal{M}} \mathrm{L}^{(2n+1)}_{\mathrm{CS}}(\mathcal{A}) - \kappa \int_{\mathcal{M}} \mathrm{L}^{(2n+1)}_{\mathrm{CS}}(\bar{\mathcal{A}}) - \kappa \int_{\partial\mathcal{M}} \mathrm{B}^{(2n)}(\mathcal{A}, \bar{\mathcal{A}}) \, . \tag{4.6}$$

If the G-bundle $\mathcal{P}$ is nontrivial, then eq. (4.6) can be written more precisely by covering $\mathcal{M}$ with local charts. This explains the introduction of the second gauge potential $\bar{\mathcal{A}}$ such that in the overlap of two charts the connections are related by a gauge transformation of the form in eq. (3.22) and the overlap contributions cancel; in this setting the coset element $g \in G/H$ is interpreted as a transition function determining the nontriviality of $\mathcal{P}$ [29].

Now we construct transgression actions for the Poincaré group using the Lagrangian of eq. (3.24) and its nonlinear representation in eq. (3.26). In this case the boundary term $\mathrm{B}^{(2n)}(\mathcal{A}, \bar{\mathcal{A}})$ defined by eq. (4.4) reads

$$\mathrm{B}^{(2n)}(\mathcal{A}, \bar{\mathcal{A}}) = n \int_0^1 \mathrm{d}t \, t^n \, \epsilon_{a_1 \cdots a_{2n+1}} R_t^{a_1 a_2} \wedge \cdots \wedge R_t^{a_{2n-3} a_{2n-2}} \wedge \omega^{a_{2n-1} a_{2n}} \wedge D_\omega \phi^{a_{2n+1}} \, . \quad (4.7)$$

Inserting eqs. (3.24, 3.26) and eq. (4.7) into eq. (4.6) we get

$$\mathrm{S}_{\mathrm{T}}^{(2n+1)} \left[\mathcal{A}, \bar{\mathcal{A}} \right] = \kappa \int_{\mathcal{M}} \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2n-1} a_{2n}} \wedge D_\omega \phi^{a_{2n+1}} \, , \quad (4.8)$$

which is a boundary term because of the Bianchi identity $D_\omega R^{ab} = 0$ and Stokes' theorem. This motivates the writing

$$\mathrm{S}^{(2n)} \left[\omega, \phi \right] = \kappa \int_{\partial \mathcal{M}} \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2n-1} a_{2n}} \phi^{a_{2n+1}} \quad (4.9)$$

as an action principle in one less dimension which corresponds to $2n$-dimensional topological Poincaré gravity. Our derivation can be regarded as a holographic principle in the sense that the transgression action in eq. (4.6) collapses to its boundary contribution once we consider gauge connections taking values in the Lie algebras associated to the linear and nonlinear realizations of the Poincaré group. The topological action of eq. (4.9) is the action of a gauged WZW model [30]; this is because the transformation law for the nonlinear gauge fields has the same form as a gauge transformation from eq. (3.22) with gauge element $g = \mathrm{e}^{-\phi^a \mathsf{P}_a} \in \frac{ISO(2n,1)}{SO(2n,1)}$ [31].

Recall that the nonlinear realization prescribes a transformation law for the field ϕ under local translations given by $\delta \phi^a = \rho^a$. This transformation breaks the symmetry of eq. (4.9) from $ISO(2n, 1)$ to $SO(2n, 1)$; this is due to the fact that the transformation law of the coset field ϕ under local translations is not a proper adjoint transformation (see eq. (3.4)).

The variation of the action in eq. (4.9) leads to the field equations

$$\epsilon_{abca_1 \cdots a_{2n-2}} D_\omega \phi^c \wedge R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2n-3} a_{2n-2}} = 0 \, , \quad (4.10)$$

$$\epsilon_{ca_1 \cdots a_{2n}} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2n-1} a_{2n}} = 0 \, . \quad (4.11)$$

Note that one can always use a gauge transformation to rotate to a frame in which $\phi^1 = \cdots = \phi^{a_{2n}} = 0$ and $\phi^{a_{2n+1}} := \phi$. This choice breaks the gauge symmetry to the residual gauge symmetry preserving the frame, which is a subgroup $SO(2n-1, 1) \hookrightarrow SO(2n, 1)$; this is just the usual Lorentz symmetry in $2n$ dimensions. If in addition one imposes the condition $\omega^{a,2n+1} = 0$ for $a = 1, \ldots, 2n$, then gauge invariance of eq. (4.9) is also preserved.

5 Topological supergravity

5.1 Three-dimensional supergravity

Supergravity in three dimensions [32, 33] can be formulated as a Chern–Simons theory for the Poincaré supergroup [34]; our spinor conventions are summarized in Appendix B. In the case of $\mathcal{N} = 1$ supersymmetry, the model is described by the action

$$\mathrm{S}^{(3)} (\mathcal{A}) = \kappa \int_{\mathcal{M}} \mathrm{L}_{\mathrm{CS}}^{(3)} (\mathcal{A}) = \kappa \int_{\mathcal{M}} \left(\epsilon_{abc} R^{ab} \wedge e^c - \mathrm{i} \bar{\psi} \wedge D_\omega \psi \right) - \frac{\kappa}{2} \int_{\partial \mathcal{M}} \epsilon_{abc} \omega^{ab} \wedge e^c \quad (5.1)$$

where ψ is a two component Majorana spinor one-form. This action is invariant (up to boundary terms) under Lorentz rotations, Poincaré translations and $\mathcal{N} = 1$ supersymmetry transformations. The gauge fields e^a, ω^{ab} and $\bar{\psi}$ transform as components of a gauge connection valued in the $\mathcal{N} = 1$ supersymmetric extension of Poincaré algebra in three dimensions given by

$$\mathcal{A} = \mathrm{i}\, e^a \, \mathsf{P}_a + \tfrac{\mathrm{i}}{2}\, \omega^{ab} \, \mathsf{J}_{ab} + \bar{\psi} \, \mathsf{Q} \ . \tag{5.2}$$

This algebra contains, in addition to the bosonic commutation relations, the supersymmetric relations

$$[\mathsf{J}_{ab}, \mathsf{Q}_\alpha] = -\tfrac{\mathrm{i}}{2}\, (\Gamma_{ab})_\alpha^{\ \beta}\, \mathsf{Q}_\beta \qquad \text{and} \qquad \{\mathsf{Q}_\alpha, \mathsf{Q}_\beta\} = (\Gamma^a)_{\alpha\beta}\, \mathsf{P}_a \ . \tag{5.3}$$

Here $\Gamma_{ab} = [\Gamma_a, \Gamma_b]$, and the set of gamma-matrices Γ_a with $a = 1, 2, 3$ defines a representation of the Clifford algebra in $2 + 1$ dimensions; then $D_\omega \psi := \mathrm{d}\psi + \tfrac{1}{4}\, \omega^{ab} \wedge \Gamma_{ab}\psi$ is the Lorentz covariant derivative in the spinor representation. Under an infinitesimal gauge transformation with parameter $\lambda = \mathrm{i}\, \rho^a \, \mathsf{P}_a + \tfrac{\mathrm{i}}{2}\, \kappa^{ab}\, \mathsf{J}_{ab} + \bar{\varepsilon}\, \mathsf{Q}$, the gauge fields transform as

$$\delta e^a = -D_\omega \rho^a + \kappa^a_{\ b}\, e^b \ ,$$
$$\delta \omega^{ab} = -D_\omega \kappa^{ab} \ ,$$
$$\delta \bar{\psi} = -D_\omega \bar{\varepsilon} - \tfrac{1}{4}\, \kappa^{ab}\, \bar{\psi}\, \Gamma_{ab} \ . \tag{5.4}$$

These transformations leave the action of eq. (5.1) invariant modulo boundary terms.

5.2 Supersymmetric SWGN formalism

The supersymmetric Stelle–West–Grignani–Nardelli formalism is treated in ref. [35] where the nonlinear realization of the supersymmetric AdS group in three dimensions is considered. Here we consider the nonlinear realization of the three-dimensional $\mathcal{N} = 1$ Poincaré supergroup [36].

Let G denote the Poincaré supergroup generated by $\{\mathsf{J}_{ab}, \mathsf{P}_a, \mathsf{Q}\}$. It is convenient to decompose G into two subgroups: The Lorentz subgroup $L = SO(2,1)$ generated by $\{\mathsf{J}_{ab}\}$ as the stability subgroup, and the Poincaré subgroup $H = ISO(2,1)$ generated by $\{\mathsf{J}_{ab}, \mathsf{P}_a\}$. We introduce a coset field associated to each generator in the coset space G/L through $\bar{\chi}\, \mathsf{Q}$ and $\phi^a \, \mathsf{P}_a$. Let us write eq. (3.2) in the form

$$g_0 \, \mathrm{e}^{-\bar{\chi}\, \mathsf{Q}} \, \mathrm{e}^{-\phi\cdot\mathsf{P}} = \mathrm{e}^{-\bar{\chi}'\, \mathsf{Q}} \, \mathrm{e}^{-\phi'\cdot\mathsf{P}} \, l_1 \tag{5.5}$$

with $l_1 \in L$. Multiplying on the right by $\mathrm{e}^{\phi\cdot\mathsf{P}}$ we get

$$g_0 \, \mathrm{e}^{-\bar{\chi}\, \mathsf{Q}} = \mathrm{e}^{-\bar{\chi}'\, \mathsf{Q}} \, h_1 \qquad \text{and} \qquad h_1 \, \mathrm{e}^{-\phi\cdot\mathsf{P}} = \mathrm{e}^{-\phi'\cdot\mathsf{P}} \, l_1 \tag{5.6}$$

with $h_1 = \mathrm{e}^{-\phi'\cdot\mathsf{P}} \, l_1 \, \mathrm{e}^{\phi\cdot\mathsf{P}} \in H$. To obtain the transformation law of the coset fields, we write these expressions in infinitesimal form

$$\mathrm{e}^{\bar{\chi}\, \mathsf{Q}} \, (g_0 - 1) \, \mathrm{e}^{-\bar{\chi}\, \mathsf{Q}} - \mathrm{e}^{\bar{\chi}\, \mathsf{Q}} \, \delta\big(\mathrm{e}^{-\bar{\chi}\, \mathsf{Q}}\big) = h_1 - 1 \ , \tag{5.7}$$

$$\mathrm{e}^{\phi\cdot\mathsf{P}} \, (h_1 - 1) \, \mathrm{e}^{-\phi\cdot\mathsf{P}} - \mathrm{e}^{\phi\cdot\mathsf{P}} \, \delta\, \mathrm{e}^{-\phi\cdot\mathsf{P}} = l_1 - 1 \ , \tag{5.8}$$

where $h_1 = h_1 \, (\bar{\chi}, \bar{\varepsilon}, \rho, \kappa)$ and $l_1 = l_1 \, (\bar{\chi}, \phi, \bar{\varepsilon}, \rho, \kappa)$. Inserting $g_0 - 1 = -\mathrm{i}\, \rho^a \, \mathsf{P}_a - \tfrac{\mathrm{i}}{2}\, \kappa^{ab}\, \mathsf{J}_{ab} - \bar{\varepsilon}\, \mathsf{Q}$, $h_1 - 1 = -\mathrm{i}\, \rho^a \, \mathsf{P}_a - \tfrac{\mathrm{i}}{2}\kappa^{ab}\, \mathsf{J}_{ab}$ and $l_1 - 1 = -\tfrac{\mathrm{i}}{2}\, \kappa^{ab}\, \mathsf{J}_{ab}$ into eqs. (5.7,5.8), we find the symmetry

transformations for the coset fields

$$\delta\phi^a = \rho^a + \tfrac{\mathrm{i}}{2}\,\bar{\varepsilon}\,\Gamma^a\chi - \kappa^a{}_c\,\phi^c\,, \tag{5.9}$$

$$\delta\bar{\chi} = \tfrac{1}{4}\,\bar{\chi}\,\kappa^{ab}\,\Gamma_{ab} + \bar{\varepsilon}\,. \tag{5.10}$$

The relations between the linear and nonlinear gauge fields can be obtained from eq. (3.22). With $g = \mathrm{e}^{-\bar{\chi}\,\mathsf{Q}}\,\mathrm{e}^{-\phi\cdot\mathsf{P}}$ we get

$$V^a = e^a - D_\omega\phi^a - \tfrac{\mathrm{i}}{2}\,D_\omega\bar{\chi}\,\Gamma^a\chi + \mathrm{i}\,\bar{\chi}\,\Gamma^a\psi\,, \tag{5.11}$$

$$W^{ab} = \omega^{ab}\,, \tag{5.12}$$

$$\bar{\Psi} = \bar{\psi} - D_\omega\bar{\chi}\,. \tag{5.13}$$

In this way the action for supergravity in three dimensions written in terms of nonlinear fields reads

$$\mathrm{S}^{(3)}\left(\bar{\mathcal{A}}\right) = \kappa\int_{\mathcal{M}}\mathrm{L}_{\mathrm{CS}}^{(3)}\left(\bar{\mathcal{A}}\right) = \kappa\int_{\mathcal{M}}\left(\epsilon_{abc}\,R^{ab}\wedge V^c - \mathrm{i}\,\bar{\Psi}\wedge D_\omega\Psi\right) - \frac{\kappa}{2}\int_{\partial\mathcal{M}}\epsilon_{abc}\,\omega^{ab}\wedge V^c\,. \tag{5.14}$$

5.3 Topological supergravity in two dimensions

In complete analogy with the bosonic case, we now construct a transgression action for the Poincaré supergroup in three dimensions. Inserting eq. (5.1) and eq. (5.14) in eq. (4.6) with

$$\mathrm{B}^{(2)}\left(\mathcal{A},\bar{\mathcal{A}}\right) = -\tfrac{1}{2}\,\epsilon_{abc}\,\omega^{ab}\wedge\left(D_\omega\phi^c + \mathrm{i}\,D_\omega\bar{\chi}\,\Gamma^c\chi - \mathrm{i}\,\bar{\chi}\,\Gamma^c\psi\right) + \mathrm{i}\,\bar{\psi}\wedge D_\omega\chi \tag{5.15}$$

we obtain

$$\mathrm{S}^{(2)}\left[\omega,\phi;\bar{\psi},\chi\right] = \kappa\int_{\partial\mathcal{M}}\left(\epsilon_{abc}\,R^{ab}\,\phi^c - 2\,\mathrm{i}\,\bar{\psi}\wedge D_\omega\chi\right)\,. \tag{5.16}$$

This action corresponds to the supersymmetric extension of topological gravity in two dimensions proposed by ref. [1]. As in the purely bosonic case, supersymmetry is broken to the Lorentz symmetry $SO(2,1)$ because of the nonlinear transformation laws in eqs. (5.9,5.10); however, the action is invariant under the full supersymmetry if one prescribes the correct transformation laws for the coset fields $\bar{\chi},\phi$ instead of considering the symmetries dictated by the nonlinear realization. The variation of the action in eq. (5.16) leads to the field equations

$$\epsilon_{abc}\left(D_\omega\phi^c - \mathrm{i}\,\bar{\psi}\,\Gamma^c\chi\right) = 0\,, \qquad D_\omega\chi = 0 = D_\omega\bar{\psi} \qquad \text{and} \qquad \epsilon_{abc}\,R^{ab} = 0\,. \tag{5.17}$$

6 WZW model for the gauged Maxwell algebra

6.1 Maxwell algebra and Chern–Simons gravity

The Maxwell algebra is a noncentral extension of the Poincaré algebra by a rank two tensor $\mathsf{Z}_{ab} = -\mathsf{Z}_{ba}$ such that

$$[\mathsf{P}_a,\mathsf{P}_b] = \mathsf{Z}_{ab} \qquad \text{and} \qquad [\mathsf{J}_{ab},\mathsf{Z}_{cd}] = \eta_{bc}\,\mathsf{Z}_{ad} + \eta_{ad}\,\mathsf{Z}_{bc} - \eta_{ac}\,\mathsf{Z}_{bd} - \eta_{bd}\,\mathsf{Z}_{ac}\,. \tag{6.1}$$

It describes the symmetries of a particle moving in an electromagnetic background [37, 38]. It is argued in ref. [39] that gauging the Maxwell algebra leads to new contributions to the

cosmological term in Einstein gravity. In this section we explore the implications of the gauged Maxwell algebra in the context of Chern–Simons gravity. In particular, we consider the three-dimensional case because it is in this dimension where Einstein gravity and Chern–Simons gravity are classically equivalent. This motivates the construction of the corresponding gauged WZW model in two dimensions.

In order to construct the Chern–Simons gravitational Lagrangians, on the one hand we need to *gauge* the Maxwell algebra, while on the other hand we need to specify the non-zero components of the invariant tensor. Gauging the Maxwell algebra is straightforward. Consider a connection one-form $\mathcal{A}$ taking values in the Maxwell algebra, which can be expanded as

$$\mathcal{A} = e^a \, \mathsf{P}_a + \tfrac{1}{2}\, \omega^{ab} \, \mathsf{J}_{ab} + \tfrac{1}{2}\, \sigma^{ab} \, \mathsf{Z}_{ab} \tag{6.2}$$

where e^a and ω^{ab} are the standard vielbein and spin connection gauge fields, and we introduce an additional rank two antisymmetric one-form $\sigma^{ab} = -\sigma^{ba}$ as the gauge field corresponding to the generator Z_{ab}. The associated invariant tensors are a little bit more involved; they can be obtained as an S-expansion starting from the AdS algebra in three dimensions. S-expansions consist of systematic Lie algebra enhancements which enlarge symmetries. They have the nice property that they provide the right invariant tensor of the expanded algebra [40], which is a key ingredient in the evaluation of Chern–Simons forms; in Appendix C we show that the Maxwell algebra can be obtained as an S-expansion of the AdS algebra. In three dimensions the resulting invariant tensors for the Maxwell algebra are found to be

$$\langle \mathsf{J}_{ab} \, \mathsf{J}_{cd} \rangle = \alpha_0 \, (\eta_{ad}\,\eta_{bc} - \eta_{ac}\,\eta_{bd}) \ , \tag{6.3}$$

$$\langle \mathsf{J}_{ab} \, \mathsf{P}_c \rangle = \alpha_1 \, \epsilon_{abc} \ , \tag{6.4}$$

$$\langle \mathsf{J}_{ab} \, \mathsf{Z}_{cd} \rangle = \alpha_2 \, (\eta_{ad}\,\eta_{bc} - \eta_{ac}\,\eta_{bd}) \ , \tag{6.5}$$

$$\langle \mathsf{P}_a \, \mathsf{P}_b \rangle = \alpha_2 \, \eta_{ab} \ , \tag{6.6}$$

where α_i, $i = 0, 1, 2$ are arbitrary constants.

With this data, one can show that the Chern–Simons gravity action for the Maxwell algebra is given by

$$\mathrm{S}_{\mathrm{CS}}^{(3)}(\mathcal{A}) = \kappa \int_{\mathcal{M}} \left(\frac{\alpha_0}{2}\, \omega^a{}_b \wedge \left(\mathrm{d}\omega^b{}_c + \frac{2}{3}\, \omega^b{}_d \wedge \omega^d{}_c \right) + \alpha_1 \, \epsilon_{abc}\, R^{ab} \wedge e^c \right. \tag{6.7}$$

$$\left. + \alpha_2 \, (T^a \wedge e_a + R^a{}_c \wedge \sigma^c{}_a) - \mathrm{d}\Big(\frac{\alpha_1}{2}\, \epsilon_{abc}\, \omega^{ab} \wedge e^c + \frac{\alpha_2}{2}\, \omega^a{}_b \wedge \sigma^b{}_a \Big) \right) ,$$

where $T^a = D_\omega e^a$ is the torsion two-form. The resulting theory contains three sectors governed by the different values of the coupling constants α_i. The first term is the gravitational Chern–Simons Lagrangian [41] while the second term is the usual Einstein–Hilbert Lagrangian. The sector proportional to α_2 contains the torsional term plus a new coupling between the gauge field σ^{ab} and the Lorentz curvature. Up to boundary terms, the action of eq. (6.7) is invariant under the local gauge transformations

$$\delta e^a = -D_\omega \rho^a + \kappa^a{}_b \, e^b \ ,$$

$$\delta \omega^{ab} = -D_\omega \kappa^{ab} \ ,$$

$$\delta \sigma^{ab} = -D_\omega \tau^{ab} - 2e^a \, \rho^b - 2\omega^a{}_c \tau^{cb} + 2\kappa^a{}_c \, \sigma^{cb} \ . \tag{6.8}$$

The variation of eq.(6.7) leads to the following equations of motion

$$\alpha_0 R_{ab} + \alpha_1 \epsilon_{abc} T^c - \alpha_2 \left(e_a \wedge e_b + \frac{1}{2} D_\omega \sigma_{ab} \right) = 0, \tag{6.9}$$

$$\alpha_1 \epsilon_{abc} R^{ab} + 2\alpha_2 T_c = 0, \tag{6.10}$$

$$\alpha_2 R_{ab} = 0. \tag{6.11}$$

Substituting eq.(6.11), with $\alpha_2 \neq 0$, into eq.(6.10) we get $T^a = 0$. Substituting again into eq.(6.9) we finally get

$$R_{ab} = 0, \tag{6.12}$$

$$T_c = 0, \tag{6.13}$$

$$D_\omega \sigma_{ab} + 2 e_a \wedge e_b = 0. \tag{6.14}$$

Thus, according to eq.(6.12, 6.13), the three dimensional Chern–Simons action for the Maxwell algebra describes a flat geometry. The new feature of the theory comes from eq.(6.14) which can be interpreted as the coupling of a matter field σ to the flat three dimensional space.

6.2   WZW model

The Maxwell group G contains the Lorentz subgroup H generated by $\{\mathsf{J}_{ab}\}$ and the coset G/H generated by $\{\mathsf{P}_a, \mathsf{Z}_{ab}\}$. Under gauge transformations, the gauge field transforms according to eq. (3.22). Let us now perform a gauge transformation with gauge element $g \in G/H$ given by

$$g = \mathrm{e}^{-\frac{1}{2} h^{ab} \mathsf{Z}_{ab}} \, \mathrm{e}^{-\phi^a \mathsf{P}_a} . \tag{6.15}$$

In terms of gauge fields, eq. (3.22) reads

$$V^a \mathsf{P}_a + \tfrac{1}{2} W^{ab} \mathsf{J}_{ab} + \tfrac{1}{2} \Sigma^{ab} \mathsf{Z}_{ab} = \mathrm{e}^{\phi^a \mathsf{P}_a} \, \mathrm{e}^{\frac{1}{2} h^{ab} \mathsf{Z}_{ab}} \, (\mathrm{d} + \mathcal{A}) \, \mathrm{e}^{-\frac{1}{2} h^{ab} \mathsf{Z}_{ab}} \, \mathrm{e}^{-\phi^a \mathsf{P}_a} \tag{6.16}$$

and it is straightforward to show using the commutation relations that

$$V^a = e^a - D_\omega \phi^a \, ,$$

$$W^{ab} = \omega^{ab} \, ,$$

$$\Sigma^{ab} = 2\phi^a e^b + \sigma^{ab} - \phi^a D_\omega \phi^b - D_\omega h^{ab} \, . \tag{6.17}$$

The final step is to compute the transgression form from eq. (4.5) with the result

$$\mathrm{B}^{(2)}(\mathcal{A}, \bar{\mathcal{A}}) = \alpha_2 \, e^a \wedge (e_a - D_\omega \phi_a) - \tfrac{\alpha_1}{2} \epsilon_{abc} \, \omega^{ab} \wedge D_\omega \phi^c$$

$$+ \tfrac{\alpha_2}{2} \, \omega^a{}_c \wedge (2\phi^c e_a - \phi^c D_\omega \phi_a - D_\omega h^c{}_a) \, . \tag{6.18}$$

The resulting action is a boundary term which corresponds to the gauged WZW action associated to the Maxwell algebra. As previously, we propose it as a Lagrangian in one less dimension

$$\mathrm{S}^{(2)}[\omega, \phi, h, e] = \kappa \int_{\partial \mathcal{M}} \left(\alpha_1 \, \epsilon_{abc} \, R^{ab} \phi^c + \alpha_2 \left(R^a{}_c \wedge h^c{}_a - e^a \wedge e_a \right) \right) . \tag{6.19}$$

This action generalizes the topological action for gravity from eq. (4.9). However, it is interesting to precise that both actions are classically equivalent, on-shell.

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The variation of eq.(6.19) gives the following equations of motion

$$\alpha_2 D_\omega h_{ab} - \alpha_1 \epsilon_{abc} D_\omega \phi^c = 0, \tag{6.20}$$

$$e_a = 0 = R_{ab}. \tag{6.21}$$

Now, making the following redefinition $\bar{\phi}^c = \alpha_2 \epsilon^{cjk} h_{jk} - 2\alpha_1 \phi^c$, it is direct to show that eq.(6.20) satisfy $\epsilon_{abc} D_\omega \bar{\phi}^c = 0$, which corresponds, together with eq.(6.21), to the field equations for the topological gravity theory in the $n = 1$ case eq.(4.10, 4.11). Note that this equivalence is only classical. It would be interesting to investigate what are the implications of this model at the quantum level.

Acknowledgments

A Poincaré-invariant Chern–Simons gravity

Poincaré gravity in $2n+1$ dimensions can be formulated as a Chern–Simons theory for the gauge group $ISO(2n, 1)$. The fundamental field is the one-form connection

$$\mathcal{A} = e^a \, \mathsf{P}_a + \tfrac{1}{2} \, \omega^{ab} \, \mathsf{J}_{ab} \tag{A.1}$$

with values in the Lie algebra $\mathfrak{iso}(2n, 1)$ whose commutation relations are given by

$$[\mathsf{J}_{ab}, \mathsf{J}_{cd}] = \eta_{ac} \, \mathsf{J}_{bd} + \eta_{bd} \, \mathsf{J}_{ca} - \eta_{bc} \, \mathsf{J}_{ad} - \eta_{ad} \, \mathsf{J}_{bc} \ ,$$

$$[\mathsf{J}_{ab}, \mathsf{P}_c] = \eta_{ac} \, \mathsf{P}_b - \eta_{bc} \, \mathsf{P}_a \ ,$$

$$[\mathsf{P}_a, \mathsf{P}_b] = 0 \ . \tag{A.2}$$

Here $\{\mathsf{J}_{ab}\}_{a,b=1}^{2n+1}$ generate the Lorentz subalgebra $\mathfrak{so}(2n, 1)$, $\{\mathsf{P}_a\}_{a=1}^{2n+1}$ generate local Poincaré translations and $(\eta_{ab}) = \mathrm{diag}\,(-1, 1, \ldots, 1)$ is a $(2n + 1)$-dimensional Minkowski metric.

In order to obtain the explicit form of the action, we use the subspace separation method [28, 42]. The subspace separation method is a systematic procedure for computing Chern–Simons forms. This mechanism is based on the extended Cartan homotopy formula [43] and has the virtue that it enables one to separate the action in terms of bulk and boundary contributions, and it splits the Lagrangian into pieces valued on the subspace structure of the gauge algebra which simplifies the calculations considerably. Following refs. [28, 42], first we decompose the gauge algebra into vector subspaces $\mathfrak{iso}\,(2n, 1) = \mathsf{V}_1 \oplus \mathsf{V}_2$ where $\mathsf{V}_1 = \mathrm{Span}_\mathbb{C}\,\{\mathsf{J}_{ab}\}$ and $\mathsf{V}_2 = \mathrm{Span}_\mathbb{C}\,\{\mathsf{P}_a\}$. Next we split the gauge potential into pieces valued in each subspace of the gauge algebra

$$\mathcal{A}_0 = 0 \ , \qquad \mathcal{A}_1 = \omega \qquad \text{and} \qquad \mathcal{A}_2 = \omega + e \tag{A.3}$$

where $\omega = \frac{1}{2}\omega^{ab}\,\mathsf{J}_{ab}$ and $e = e^a\,\mathsf{P}_a$. Computing each component of the triangle equation of eq. (4.2) we find

$$Q^{(2n+1)}_{\mathcal{A}_2\leftarrow\mathcal{A}_1} = \epsilon_{a_1\cdots a_{2n+1}}\,R^{a_1 a_2}\wedge\cdots\wedge R^{a_{2n-1}a_{2n}}\wedge e^{a_{2n+1}}\ , \tag{A.4}$$

$$Q^{(2n+1)}_{\mathcal{A}_1\leftarrow\mathcal{A}_0} = 0\ , \tag{A.5}$$

$$Q^{(2n)}_{\mathcal{A}_2\leftarrow\mathcal{A}_1\leftarrow\mathcal{A}_0} = -n\int_0^1 \mathrm{d}t\ t^n\,\epsilon_{a_1\cdots a_{2n+1}}\,R_t^{a_1 a_2}\wedge\cdots\wedge R_t^{a_{2n-3}a_{2n-2}}\wedge\omega^{a_{2n-1}a_{2n}}\wedge e^{a_{2n+1}}\ . \tag{A.6}$$

Here we have used the fact that the only nonvanishing components of the invariant tensor for the Poincaré algebra are given by

$$\left\langle\mathsf{J}_{a_1 a_2}\cdots\mathsf{J}_{a_{2n-1}a_{2n}}\,\mathsf{P}_{a_{2n+1}}\right\rangle = \frac{2^n}{n+1}\,\epsilon_{a_1\cdots a_{2n+1}}\ . \tag{A.7}$$

Inserting eqs. (A.4–A.6) into eq. (4.2) and using $\mathrm{L}_{\mathrm{CS}}^{(2n+1)}(\mathcal{A}) = Q^{(2n+1)}_{\mathcal{A}_2\leftarrow\mathcal{A}_0}$ we obtain eq. (3.24).

B Spinors in three dimensions

B.1 Gamma-matrices

In this appendix we summarise our conventions regarding the Clifford algebra and spinors in three dimensions, following ref. [44]. The Clifford algebra is defined by

$$\Gamma_a\,\Gamma_b + \Gamma_b\,\Gamma_a = 2\eta_{ab} \tag{B.1}$$

where $(\eta_{ab}) = \mathrm{diag}(-1,1,1)$ and the minus sign is along the timelike direction. The Pauli spin matrices σ_a with $a = 1,2,3$ provide a representation of the Clifford algebra with signature $(0,3)$. Since we are interested in fixing a representation with signature $(1,2)$, all we need to do is to multiply one of the Pauli matrices by the imaginary unit and declare it to be Γ_1. An explicit representation is then given by

$$\Gamma_1 = \mathrm{i}\,\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ ,\qquad \Gamma_2 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\qquad\text{and}\qquad \Gamma_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{B.2}$$

There always exists a charge conjugation matrix $\mathcal{C}$ which in three dimensions satisfies

$$\mathcal{C}^\top = -\mathcal{C}\qquad\text{and}\qquad \Gamma_a^\top = -\mathcal{C}\,\Gamma_a\,\mathcal{C}^{-1}\ . \tag{B.3}$$

In the chosen basis of gamma-matrices it can be taken to be $\mathcal{C} = \sigma_2 = -\mathrm{i}\,\Gamma_1$. The charge conjugation matrix satisfies $\mathcal{C} = \mathcal{C}^{-1} = \mathcal{C}^\dagger$.

B.2 Majorana spinors

The minimal irreducible spinor in three dimensions is a two real component Majorana spinor. Every Majorana spinor satisfies a reality condition which can be established by demanding that the Majorana conjugate equals the Dirac conjugate

$$\bar\psi := \psi^\top\mathcal{C} = -\mathrm{i}\,\psi^\top\Gamma_1\ . \tag{B.4}$$

Spinors carry indices ψ_α and gamma-matrices act on them in such a way that $\Gamma_a\psi := (\Gamma_a)^\alpha{}_\beta\, \psi_\alpha$. In order to raise and lower indices, we introduce matrices $(\mathcal{C}^{\alpha\beta})$, $(\mathcal{C}_{\alpha\beta})$ related to the charge conjugation matrix, and we use the convention of raising and lowering indices according to the NorthWest–SouthEast convention ($\searrow$). This means that the position of the indices should appear in that relative position as

$$\psi^\alpha = \mathcal{C}^{\alpha\beta}\,\psi_\beta \qquad \text{and} \qquad \psi_\alpha = \psi^\beta\,\mathcal{C}_{\alpha\beta}\ , \tag{B.5}$$

which implies that

$$\mathcal{C}^{\alpha\beta}\,\mathcal{C}_{\gamma\beta} = \delta^\alpha_\gamma \qquad \text{and} \qquad \mathcal{C}_{\beta\alpha}\,\mathcal{C}^{\beta\gamma} = \delta^\gamma_\alpha\ . \tag{B.6}$$

We choose the identifications in such a way that the Majorana conjugate $\bar\psi$ is written as ψ^α. Comparing eq. (B.4) with eq. (B.5), one then finds $(\mathcal{C}^{\alpha\beta}) = \mathcal{C}^\top$ and $(\mathcal{C}_{\alpha\beta}) = \mathcal{C}^{-1}$.

C Maxwell algebra by S-expansion

C.1 S-expansions of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra and $S = \{\lambda_\alpha\}$ a finite abelian semigroup with composition law $\lambda_\alpha \cdot \lambda_\beta$. By [40, Theorem 3.1] the direct product $S \times \mathfrak{g}$ is also a Lie algebra. There are cases in which it is possible to systematically extract Lie subalgebras from $S \times \mathfrak{g}$. For instance, we can start by decomposing $\mathfrak{g}$ into a direct sum of subspaces $\mathfrak{g} = \bigoplus_{p\in I} V_p$ where I is some index set. The Lie algebra structure of $\mathfrak{g}$ can be encoded in subsets $I_{(p,q)} \subset I$ according to $[V_p, V_q] \subset \bigoplus_{r\in I_{(p,q)}} V_r$. If the semigroup S also admits a decomposition into subsets $S = \bigcup_{p\in I} S_p$ satisfying $S_p \cdot S_q \subset \bigcap_{r\in I_{(p,q)}} S_r$, we say that the algebra and the semigroup decompositions are in *resonance*. Then $\mathfrak{g}_{\mathrm{R}} := \bigoplus_{p\in I} S_p \times V_p$ is a "resonant subalgebra" of $S \times \mathfrak{g}$ [40, Theorem 4.2].

If one further has a *zero* element in the semigroup, i.e., an element $0_{\mathrm{S}} \in S$ such that $0_{\mathrm{S}} \cdot \lambda_\alpha = 0_{\mathrm{S}}$ for all $\lambda_\alpha \in S$, then the whole sector $0_{\mathrm{S}} \times \mathfrak{g}$ can be removed from the resonant subalgebra by imposing $0_{\mathrm{S}} \times \mathfrak{g} = 0$. The remaining structure, which we refer to as the 0_{S}-reduced algebra, is still a Lie algebra [40, Theorem 6.1].

C.2 S-expansion of the AdS algebra

We now show that the Maxwell algebra can be obtained by an S-expansion of the AdS algebra. Let $S_{\mathrm{E}}^{(2)}$ be the semigroup [45]

$$S_{\mathrm{E}}^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\} \tag{C.1}$$

with composition law

$$\lambda_\alpha \cdot \lambda_\beta := \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha + \beta \le 3\ , \\ \lambda_3 & \text{if } \alpha + \beta > 3\ . \end{cases} \tag{C.2}$$

Recall that the AdS algebra $\mathfrak{g} = \mathfrak{so}(d-1,2)$ in d dimensions is given by

$$\left[\, \bar{\mathsf{J}}_{ab}, \bar{\mathsf{J}}_{cd} \,\right] = \eta_{bc}\, \bar{\mathsf{J}}_{ad} + \eta_{ad}\, \bar{\mathsf{J}}_{bc} - \eta_{ac}\, \bar{\mathsf{J}}_{bd} - \eta_{bd}\, \bar{\mathsf{J}}_{ac}\ , \tag{C.3}$$

$$\left[\, \bar{\mathsf{J}}_{ab}, \bar{\mathsf{P}}_c \,\right] = \eta_{bc}\, \bar{\mathsf{P}}_a - \eta_{ac}\, \bar{\mathsf{P}}_b\ , \tag{C.4}$$

$$\left[\, \bar{\mathsf{P}}_a, \bar{\mathsf{P}}_b \,\right] = \bar{\mathsf{J}}_{ab}\ . \tag{C.5}$$

This algebra can be decomposed into two subspaces $\mathfrak{g} = V_0 \oplus V_1$ where $V_0 = \mathrm{Span}_{\mathbb{C}} \{\bar{J}_{ab}\}$ and $V_1 = \mathrm{Span}_{\mathbb{C}} \{\bar{P}_a\}$. In terms of these subspaces, the AdS algebra has the structure

$$[V_0, V_0] \subset V_0 \;, \qquad [V_0, V_1] \subset V_1 \qquad \text{and} \qquad [V_1, V_1] \subset V_0 \;. \tag{C.6}$$

If we now choose the partition for the semigroup $S_E^{(2)}$ given by

$$S_0 = \{\lambda_0, \lambda_2\} \cup \{\lambda_3\} \qquad \text{and} \qquad S_1 = \{\lambda_1\} \cup \{\lambda_3\} \;, \tag{C.7}$$

then this partition is resonant with respect to the structure of the AdS algebra: Under the semigroup multiplication law we have

$$S_0 \cdot S_0 \subset S_0 \;, \qquad S_0 \cdot S_1 \subset S_1 \qquad \text{and} \qquad S_1 \cdot S_1 \subset S_0 \tag{C.8}$$

which agrees with the decomposition in eq. (C.6). The resonance condition allows us to construct a resonant subalgebra $\mathfrak{g}_R$ defined by

$$\mathfrak{g}_R = W_0 \oplus W_1 := (S_0 \times V_0) \oplus (S_1 \times V_1) \;. \tag{C.9}$$

Explicitly one has

$$W_0 = \{\lambda_0, \lambda_2, \lambda_3\} \times \mathrm{Span}_{\mathbb{C}} \{\bar{J}_{ab}\} =: \mathrm{Span}_{\mathbb{C}} \{J_{ab,0}, J_{ab,2}, J_{ab,3}\} \;,$$
$$W_1 = \{\lambda_1, \lambda_3\} \times \mathrm{Span}_{\mathbb{C}} \{\bar{P}_a\} =: \mathrm{Span}_{\mathbb{C}} \{P_{a,1}, P_{a,3}\} \;. \tag{C.10}$$

Since λ_3 is a zero element in the semigroup, one can extract another subalgebra by setting $J_{ab,3} = P_{a,3} = 0$; this choice still preserves the Lie algebra structure of the residual algebra. This algebra is called a 0-forced resonant algebra and therefore we are left with the subspaces

$$\tilde{W}_0 = \mathrm{Span}_{\mathbb{C}} \{J_{ab,0}, J_{ab,2}\} \qquad \text{and} \qquad \tilde{W}_1 = \mathrm{Span}_{\mathbb{C}} \{P_{a,1}\} \;. \tag{C.11}$$

In order to obtain a presentation for the 0-forced resonant algebra we use eqs. (C.3–C.5) together with eq. (C.2) to compute the commutation relations and identify

$$J_{ab} := J_{ab,0} \;, \qquad Z_{ab} := J_{ab,2} \qquad \text{and} \qquad P_a := P_{a,1} \tag{C.12}$$

to obtain the Maxwell algebra in d dimensions

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} \;,$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} \;,$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \;,$$
$$[P_a, P_b] = Z_{ab} \;,$$
$$[Z_{ab}, Z_{cd}] = 0 = [Z_{ab}, P_c] \;. \tag{C.13}$$

C.3 Invariant tensors

The S-expansion procedure also provides the invariant tensors associated to the expanded algebra; here we study the particular case of $d = 3$ dimensions. The invariant tensors of the AdS

algebra $\mathfrak{so}(2,2)$ are given by [46]

$$\left\langle\, \bar{\mathsf{J}}_{ab}\, \bar{\mathsf{J}}_{cd}\, \right\rangle = \mu_0\, (\eta_{ad}\,\eta_{bc} - \eta_{ac}\,\eta_{bd})\ ,$$

$$\left\langle\, \bar{\mathsf{J}}_{ab}\, \bar{\mathsf{P}}_c\, \right\rangle = \mu_1\, \epsilon_{abc}\ ,$$

$$\left\langle\, \bar{\mathsf{P}}_a\, \bar{\mathsf{P}}_b\, \right\rangle = \mu_0\, \eta_{ab}\ , \tag{C.14}$$

where μ_i, $i = 0, 1$ are arbitrary constants. By [40, Theorem 7.2], the S-expanded tensors are given by the formula

$$\left\langle \mathsf{T}_{A,\alpha}\, \mathsf{T}_{B,\beta} \right\rangle = \tilde{\alpha}_\gamma\, K_{\alpha\beta}{}^{\gamma}\, \left\langle \mathsf{T}_A\, \mathsf{T}_B \right\rangle \tag{C.15}$$

where $\tilde{\alpha}_\gamma$ are also arbitrary constants, and $K_{\alpha\beta}{}^{\gamma}$ is called a K-two selector which is a function with values 1 if $\gamma = \gamma(\alpha\,\beta)$ according to the semigroup multiplication law and 0 otherwise. The application of the formula in eq. (C.15) for the S-expanded generators $\mathsf{J}_{ab,0}$, $\mathsf{J}_{ab,2}$ and $\mathsf{P}_{a,1}$ gives the invariant tensors for the Maxwell algebra in eqs. (6.3–6.6), with the redefined constants

$$\alpha_0 := \tilde{\alpha}_0\,\mu_0\ , \qquad \alpha_1 := \tilde{\alpha}_1\,\mu_1 \qquad \text{and} \qquad \alpha_2 := \tilde{\alpha}_2\,\mu_0\ . \tag{C.16}$$

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