

RESEARCH ARTICLE

A first-digit anomaly in the 2009 Iranian presidential election

Boudewijn F. Roukema

*Toruń Centre for Astronomy, Faculty of Physics, Astronomy and Informatics, Nicolaus
Copernicus University, ul. Gagarina 11, 87-100 Toruń, Poland**(received date: ...)*

A local bootstrap method is proposed for the analysis of electoral vote-count first-digit frequencies, complementing the Benford's Law limit. The method is calibrated on five presidential-election first rounds (2002–2006) and applied to the 2009 Iranian presidential-election first round. Candidate K has a highly significant ($p < 0.15\%$) excess of vote counts starting with the digit 7. This leads to other anomalies, two of which are individually significant at $p \sim 0.1\%$, and one at $p \sim 1\%$. Independently, Iranian pre-election opinion polls significantly reject the official results unless the five polls favouring candidate A are considered alone. If the latter represent normalised data and a linear, least-squares, equal-weighted fit is used, then either candidates R and K suffered a sudden, dramatic ($70\% \pm 15\%$) loss of electoral support just prior to the election, or the official results are rejected ($p \sim 0.01\%$).

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1. Introduction

The results of the 12 June 2009 presidential-election first round held in the Islamic Republic of Iran are of high political importance in Iran. International interest in these results is also considerable. On 14 June 2009, the Ministry of the Interior (MOI) published a table of the numbers of votes v_{ij} received by the j -th of the $n = 4$ candidates for the i -th of $m = 366$ voting areas [23]. In order to avoid focussing on personalities, the four candidates will be referred to here as A, R, K, and M, following the order given in the table. These letters correspond to the conventional Roman alphabet transliteration of the four candidates' names by which they are frequently referred to. The total votes $y_j := \sum_i v_{ij}$ for these four candidates from the MOI table give A as the winner with 24,515,209 votes, against R with 659,281 votes, K with 328,979 votes, and M with 13,225,330 votes. A second voting round was not held.

The total numbers of votes per voting area $x_i \geq \sum_j v_{ij}$ (invalid votes are included in x_i) in the MOI's data vary from about 10^4 to 10^6 , i.e. by two orders of magnitude (powers of 10). This suggests that Benford's Law (Newcomb 27; Benford 8) may be applicable, i.e. it may be useful to test the null hypothesis that the first digit in the candidates' absolute numbers of votes, represented in the decimal system, are consistent with random selection from a uniform, base 10 logarithmic distribution modulo 1. This test has been historically proposed for finding hints of artificial interference in statistical data sets. The reason is that common intuition suggests that the frequencies of occurrence of the first digit (using the decimal system) in a

large data set of empirical data that cover at least an order of magnitude should be approximately equally distributed between the nine non-zero digits, i.e. about $1/9 \approx 11\%$ of the first digits should be 1, about 11% should be 2, $\dots$, and about 11% should be 9. Benford's Law contradicts this. For example, the most striking characteristic of Benford's Law follows from Eq. (1) below: the first digit is 1 with a frequency of $\log_{10} 2 \approx 30\%$, i.e. it occurs much more frequently than any other digit, with a much higher frequency than 11%. The frequencies of other digits than the first digit can also be analysed with Benford's Law. Independently of the present analysis, the distributions of the *second* digit were analysed for this same data set [19] and the *last* digits of a related data set also released by the MOI were analysed [6] using a method developed earlier using Nigerian electoral data [7].

The decimal first digit version of Benford's Law [8, 27] can be given informally by stating that for many real world samples of values that span several orders of magnitude, the relative frequency of the occurrence of digit $d \in \{1, \dots, 9\}$ as the first digit in decimal representations of real numbers tends towards

$$f(d) = \log_{10} \left(1 + \frac{1}{d} \right) \quad (1)$$

as the sample size and logarithmic distribution width increase. Equation (1) should be a good approximation if a sample can be expected to be drawn from a probability distribution of a random variable X that varies slowly enough over several orders of magnitude in such a way that

$$Y := \log_{10} X - \lfloor \log_{10} X \rfloor, \quad (2)$$

i.e. the *folding* of X to a single decade, is approximately uniform, where $\lfloor x \rfloor$ is the greatest integer $\leq x$. Because the logarithmic scale is folded into a single decade, statistical populations do not necessarily need to span many orders of magnitude in order to approximately satisfy Benford's Law. A folded logarithmic distribution (for a fixed base) will tend towards a uniform distribution as the logarithmic variance of the unfolded distribution increases. However, it is not clear how fast and smooth this convergence occurs. For example, a logarithmically uniform distribution on $z_1 \leq \log_{10} X < z_2$ will not fold to a uniform distribution unless $z_2 - z_1$ is an integer. Nevertheless, historically, Benford's Law has been found to apply well in practice in many cases, especially when several different distributions are combined to constitute a single distribution.

A detailed justification and generalisation of several equivalent formulations of Benford's Law are given by Hill, who refers to Benford's Law as the “central-limit-like theorem for significant digits” [16]. To what degree should Benford's Law apply to the Iranian MOI data set for the 2009 presidential election first round of voting? To the extent that the total voting populations x_i of the voting areas used for the MOI data set constitute a mix of many processes—the growth of towns and cities over thousands of years—that can be modelled as if they were a mix of several different (mostly non-uniform) random processes corresponding to a mix of probability distributions, including the still rapid (over 1%) population growth and the freedom to vote anywhere in Iran independently of one's place of residence, these total votes x_i may to some degree satisfy base-neutrality and/or scale-neutrality. [See Definition 5 of [16] for definitions of base and scale bias.] On the other hand, political and administrative effects might introduce strong base or scale biases.

However, rather than x_i , what are even more likely to constitute a mix of (mostly

non-uniform) random processes are the processes leading to the fractional voting rates $\widetilde{v}_{ij} := v_{ij}/x_i \leq 1$ for individual candidates for voting areas of a given population size x_i , where approximately equal x_i 's are grouped together at x . If (i) *the voting rates $\widetilde{v}_{ij}$ vary by a large fraction of an order of magnitude near $x_i \approx x$* , then the votes v_{ij} in the voting areas of size $x_i \approx x$ should also scatter by a large fraction of an order of magnitude. If, in addition, (ii) *the distribution of $\log_{10} \widetilde{v}_{ij}(x_i \approx x)$ varies slowly enough with $\log_{10} x$* , then it should be possible that scale-neutrality is satisfied well enough for Benford's Law to be applicable.

Iranian society has a continuous and complex urban history dating back about 6000 years. Hence, at least as much as in any other complex society, Iranian voting patterns can be expected to be the result of a highly rich mix of numerous social, political, economic, and historical factors that can influence voting decisions, as well as natural environmental factors (e.g. the weather [2, 14]) and the cognitive margin of freedom that individuals retain with respect to these broader factors. Are all these factors enough to satisfy the conditions (i) and (ii) sufficiently well such that the MOI data set can be statistically tested using Benford's Law for the vote counts v_{ij} ?

These qualitative descriptions of the two conditions hide a lot of quantitative detail, which could require as many as six parameters to quantify. For example, these could include, respectively, quantitative definitions of (i) "vary" in terms of standard deviations, "large", and " $\approx$ ", and of (ii) at least two parameters to characterise the distribution of $\log_{10} \widetilde{v}_{ij}$, and "slowly". Clearly, it would be preferable to use a non-parametric alternative. Moreover, rather than attempt a theoretical proof of whether or not a formalisation of these conditions would imply Benford's Law, it would be preferable to formalise the conditions in such a way that they define a statistical model of the data that can be used directly, independently of how closely the data can be expected to approach the limiting form of Benford's Law.

Hence, here, a statistical model is defined using a local bootstrap method, motivated by conditions (i) and (ii). This is presented in Section 2.2. This method is non-parametric in terms of the data set being tested, but uses one fixed, base-related parameter. The aim of the "local" property of the method is to increase the method's specificity (reduce the chance that it falsely rejects the null hypothesis that no artificial interference in the data has taken place), at the cost of decreasing the method's power (the chance that the method misses genuine anomalies is increased). Bootstrap methods can give biased and/or skewed estimates of confidence intervals, so the probabilities inferred using the method are calibrated against several recent presidential-election first rounds prior to the 2009 Iranian presidential-election first round. These data sets and the Iranian 2009 MOI vote counts are described in Section 2.3. The first digit frequencies in the Iranian 2009 MOI data are examined using this calibrated probability distribution.

In addition to the internal statistical test provided by first-digit frequencies, external tests of the MOI data can be made by comparing these data to a compilation of pre-election opinion polls that were openly and transparently documented by the English-language community prior to the election day. This compilation of polls is described in Section 2.4.

The calibration of the local bootstrap simulations using the earlier presidential elections is presented in Section 3.1. The first-digit frequencies of the four candidates' official vote counts and their raw and calibrated confidence levels are presented in Section 3.2 and Figs 10–13. The internal consistency among the pre-election opinion polls and their consistency with the MOI data set are presented in Section 3.3. Discussion, including several anomalies that follow from the basic

result, is given in Section 4. A summary and conclusion are presented in Section 5.

2. Method

2.1 Data and model files

The data from [23] used in this analysis and the data from the French (2002, 2007), Iranian (2005), Polish (2005) and Brazilian (2006) presidential-election first rounds are listed in the text file PRES_ELECTIONS. An OCTAVE script¹ is provided as a plain text file BENFORD_ELECTIONS.M for carrying out the analysis in this paper, using the input file PRES_ELECTIONS. Both files are part of the preprint version of this article at ArXiv.org.²

2.2 Empirical, local bootstrap model of presidential election vote counts

An empirical, local bootstrap model of presidential election vote counts is defined as follows.

2.2.1 Local bootstrap

For the 2009 Iranian presidential election or another similar presidential election, let the numbers of votes received by the j -th of the n candidates for the i -th of m voting areas be v_{ij} . Let the total numbers of votes per voting area be $x_i := \sum_{j=1}^{n+1} v_{ij}$ (where $v_{i(n+1)}$ represents invalid votes if known, and otherwise is set to zero), the total votes per candidate be $y_j := \sum_i v_{ij}$, and the normalised voting rates (estimates of a candidate's electoral popularity in a given area) be $\widetilde{v}_{ij} := v_{ij}/x_i \leq 1$.

Sort the x_i such that the sequence $x_{i_1}, x_{i_2}, \dots, x_{i_m}$ is (w.l.o.g.) in ascending order.

DEFINITION 2.1 *A local bootstrap realisation for candidate j is a set of simulated votes*

$$\{\widetilde{v}_{i_{k'}j} x_{i_k}\}_{k=1,\dots,m} \quad (3)$$

where the values $k'(k)$ are drawn randomly from G , a realisation of a Gaussian probability density function of width $m\Delta$, where $\Delta := \log_{10}(10/9) \approx 0.0458$, and centred at k , truncated at the limits of $\{k\}$, i.e.

$$k'(k) := \max(1, \min(m, \lfloor G(k, m\Delta) + 0.5 \rfloor)). \quad (4)$$

The meaning of $\widetilde{v}_{i_{k'}j}$ can be thought of by starting with an extremely small value of Δ , rather than the defined value. For example, $\Delta = 10^{-100}$ would in practice make all realisations of $G(k, m\Delta)$ on a 2012 computer yield $G = k$, so that, effectively, $k'(k) \equiv k$ and $\widetilde{v}_{i_{k'}j} \equiv \widetilde{v}_{ikj} := v_{ikj}/x_{i_k}$, giving a realisation identical to the empirical data set. Increasing Δ to 0.0458 allows a bootstrap from voting areas with the index k' slightly lower or higher than k , i.e. a “real” voting rate for candidate j is selected from a voting area with a slightly lower or higher (or possibly the same) total voting population. Thus, this is a “local” bootstrap—the simulated data are generated using the empirical data directly, but instead of the standard

¹<http://www.gnu.org/software/octave>

²<http://arXiv.org/e-print/0906.2789>

bootstrap method, the random selection process consists of just a slight “nudge” to the original data among voting areas of about the same voting population size.

If the x_{i_k} are distributed log-uniformly over one decade, i.e. if $\log_{10} x_{i_k}$ is distributed log-uniformly within the decade from z_1 to $10z_1$ for some z_1 , then the fraction of x_{i_k} values contained within an interval in k of width $m\Delta$ is $(m\Delta)/m = \Delta$. That is, in this case, the smoothing scale in x_{i_k} is that of the narrowest first-digit interval, $\Delta := \log_{10}(10/9)$. Narrower and wider x_{i_k} distributions will be less and more smoothed, respectively.

A full bootstrap realisation over the set of voting rates $\widetilde{v_{ij}}$ for a given candidate j would be equivalent to assuming that the probability distributions for voting rates in low and high population voting areas are independent and identically distributed, which is unlikely to be realistic. Apart from some correlation between voting area size x_{i_k} and various sociological characteristics, the Poisson nature of many processes should modify the variance of the voting rate as a function of x_{i_k} .

If Definition 2.1 is extended to the limit of $\Delta \rightarrow 0$, i.e. if the “local” aspect of the definition is taken to its limit, then the realisations approach an exact reproduction of the empirical data set itself. For a value of Δ that is small, but not so small as to mostly generate exact reproductions of the data, the realisations should sample the distribution of voting rates $\widetilde{v_{i_k,j}}$ near x_{i_k} . That is, any vote realisation $\{\widetilde{v_{i_k,j}}x_{i_k}\}_{|k-k_0|<m\Delta}$ should approximate the actual distribution $\{v_{i_k,j}\}_{|k-k_0|<m\Delta}$ near any $x_{i_{k_0}}$. Hence, within the smallest of the nine logarithmic intervals relevant for decimal first digit counts, the realisations should (a) (re-)sample the empirical distribution of voting rates near x_{i_k} , whether or not the variance is large or small, and (b) should approximately mimic the changing properties of this distribution as x_{i_k} decreases or increases. That is, the realisations should formalise the conditions (i) and (ii) presented in Section 1. To the degree that these two conditions are satisfied, Benford’s Law is likely to be a good approximation. However, the first-digit frequency distributions inferred from the local bootstrap realisations can be used independently of the degree to which the two conditions are satisfied, i.e., without assuming the Benford’s Law limiting case.

Thus, the voting data x_i and v_{ij} themselves are used to construct discrete probability distributions from which simulated samples of the set of votes are generated using random resampling allowing repeats. The simulated samples are used to estimate the confidence intervals for first-digit counts. The method is “local” in the sense that each bootstrap is performed within a small subset of the data near a given total vote x_{i_k} , so that the simulated data set should very closely match the empirical data set. The method is non-parametric in terms of the data set, and has a fixed, base-related dependence through Δ .

For these reasons, the method is conservative. By using the data to model itself, some types of artificial interference in the data may be mimicked in the simulations, so that the interference would not be detected. In contrast, any statistically significant difference between the simulations and the data would imply that the data are unusually sensitive to resampling the voting rates $\widetilde{v_{ij}}$ for voting areas of approximately the same voting population x .

For example, suppose that about 20% of voters in voting areas of voting population $x \approx 80,000$ typically voted for candidate j , but this varied down to about 10% and up to about 40%, i.e. from 8,000 to 32,000 people voted for candidate j in voting areas that have about 80,000 voters, with 16,000 being a typical vote for candidate j . The local bootstrap model proposed here assumes that reselecting these percentages randomly from within the same range, 8,000 to 32,000, for this same set of voting areas, and applying the same process for each x , should not affect the overall statistical results. The model does not assume any particular

shape of the distribution of v_{ij} in the range from 8,000 to 32,000, e.g. normal or log-normal, unimodal or multimodal, nor does it assume any particular skewness or kurtosis. It only assumes that the distribution should statistically match the observed distribution of the voting rates $\widetilde{v_{i_k j}}$ within this limited subset of voting areas.

If outliers (rare extreme values, e.g. 800 votes for candidate j in one of the voting areas of 80,000 voters in the example just given) exist among the $\widetilde{v_{ij}}$ of the empirical data set, then they will also be included in the simulated data sets with the same frequencies (in the limit of many realisations). This is an example of a potentially suspect feature of the empirical data set—outliers—that would not be detected to be significantly unusual by this method. Hence, it is difficult to see how randomly reassigning votes in a statistically identical way among these voting areas should destroy any statistical properties of the votes, such as their decimal first-digit frequencies.

2.2.2 Empirical calibration

Using candidates in presidential elections other than the 2009 Iranian election first round, as detailed in Section 2.3.2, local bootstrap realisations using Definition 2.1 are realised for each candidate j , except for candidates whose votes are too low. Candidates are considered to have vote counts that are too low if 1% or more of their vote counts satisfy $v_{ij} \leq 1$. A vote count of zero has an undefined logarithm, and vote counts below 10 are likely to create strong discreteness effects. Hill's formalisation of Benford's Law strictly applies only to random variables on the set of positive reals, not to a random variable on non-negative integers [16]. Hence, low vote counts are likely to decrease the degree to which Benford's-Law-like behaviour occurs. Candidate 10 in the Polish 2005 first-round election and candidates 1 and 4 in the Brazilian 2006 first-round election are excluded from the analysis using this definition of "too low". This leaves 51 candidates from the five elections. None of the Iranian 2009 election first-round candidates are excluded by this criterion.

From an ensemble of local bootstrap realisations for a given candidate j in a given election, let the relative frequency with which the frequency of first digit d_1 in a simulated vote count $\{\widetilde{v_{i_k j}} x_{i_k}\}_{k=1, \dots, m}$ is found to be below the frequency of d_1 in the candidate's empirical vote count $\{v_{ij}\}_{i=1, \dots, m}$ be $c_{bj}(d_1)$. This corresponds to an upper, one-sided, bootstrap-estimated confidence level when $c_{bj} > 0.5$.

Bootstrap methods in general can require corrections for bias and skewness. One recommended way to correct for this in the absence of external empirical data is the bias-corrected, accelerated method. However, for small samples this is not always reliable in practice [34]. Moreover, in the present case, we have several empirical data sets that are publicly available for analysis, are independent of the data set of interest, and should represent similar sociological processes. Hence, the approach adopted here is to use these empirical data sets to calibrate a correction for bias and skewness.

The bootstrap confidence levels $c_{bj}(d)$ obtained for the 9 non-zero digits for the 51 candidates constitute a set of $n_b = 459$ values of c_b . If the local bootstrap method is accurate in estimating c_b , then the frequency distribution of c_b should be approximated by $P(c < c_b) = c_b$. If this is not the case, then the actual frequency distribution of c_b over the whole control data set of 459 values can be used to estimate an *empirical* set of confidence levels c_e . This implicitly assumes that the 459 values are independently drawn from a single probability distribution. This is clearly only an approximation, depending on the degree to which the number of digits (9), the numbers of candidates per election (5-16, where low vote candidates have been excluded as stated above), and the number of elections (5) are

high enough. None of these three numbers is individually high, but putting them together, $n_b = 459$ may be high enough for this to be a sufficient approximation. Provided that the function $c_e(c_b)$ is smooth enough over the range of interest, this can be used to convert individual c_b estimates in the data set of interest to more accurate confidence levels $c_e(c_b)$. This is the approach adopted here.

The most important regions of the function $c_e(c_b)$ are those near 0 and 1. In particular, any fit $c_e(c_b)$ must satisfy $c_e(0) = 0$ and $c_e(1) = 1$. Let us define

$$\begin{aligned}\alpha_b^- &:= 1 - c_b, \quad c_b < c_b^* \\ \alpha_b^+ &:= c_b, \quad c_b \geq c_b^*,\end{aligned}\tag{5}$$

where c_b^* is the median value of c_b . The n_b values of c_b are sorted in ascending order and each k -th value is paired with

$$c_e(k) := (k - 0.5)/n_b,\tag{6}$$

so that

$$\begin{aligned}\alpha_e^- &:= 1 - c_e, \quad c_b < c_b^* \\ \alpha_e^+ &:= c_e, \quad c_b \geq c_b^*\end{aligned}\tag{7}$$

can be defined. Hence, an almost-everywhere smooth, continuous, piecewise fit $c_e(c_b)$ allowing one free parameter for each part of $c_e(c_b)$ is obtained by (non-linear) least squares fitting $2\alpha_e^-$ against $(\alpha_b^-/c_b^*)^{\beta^-}$ and $2\alpha_e^+$ against $[\alpha_b^+/(1 - c_b^*)]^{\beta^+}$, over the intervals $c_b < c_b^*$ and $c_b \geq c_b^*$, respectively, where β^- and β^+ are free parameters. The fitted solution is

$$c_e(c_b) = \begin{cases} 0.5 \left(\frac{c_b}{c_b^*} \right)^{\beta^-}, & c_b < c_b^* \\ 1 - 0.5 \left(\frac{1 - c_b}{1 - c_b^*} \right)^{\beta^+}, & c_b \geq c_b^*. \end{cases}\tag{8}$$

This function necessarily satisfies $c_e(0) = 0$ and $c_e(1) = 1$ and is continuous through $c_b = c_b^*$, where $c_e(c_b^*) = 0.5$. The first derivative dc_e/dc_b is not necessarily continuous at $c_b = c_b^*$, but if the values of c_b are themselves distributed smoothly, then the derivative should not change abruptly. The non-linear least squares fitting method used in this work to calculate β^- and β^+ (Sect. 3.1, Fig. 8) is iterative. The power β^- (or β^+) is initially assumed to lie in the range (0.2, 5.0). Ten uniformly spaced values in this range are tested and the minimum of χ^2 is used to choose the best estimate for that iteration. Successive iterations halve the width of the range, centre it on the previous best fit, and test ten values in the new, smaller range. Fifty iterations easily converge and give the accuracy required.

2.3 Data sets

2.3.1 Iranian Ministry of the Interior 14 June 2009 data set

The primary data of interest here consist of the data set published by the Iranian Ministry of the Interior (MOI) data on 14 June 2009 [23]. The data set was

archived the same day at the URL <http://www.webcitation.org/5hXHfYNbN>.¹ A plain text form of this data, along with data from the calibration set of presidential elections, is provided as the file PRES_ELECTIONS (see Sect. 2.1). The following day, the MOI published a new version of the data with minor changes. The largest obvious modification was that the total number of votes in Bandar-Abbas in the 14 June file was 390,141, exactly 100,000 votes greater than the sum of the individual votes and invalid votes, and greater than the 2005 Bandar-Abbas population of about 350,000, while in the 15 June file, this was corrected to 290,141. Here, the original 14 June file is used. These two XLS format files use Western Arabic (European) numerals. A third file, in PDF format and using Arabic-Indic (not Persian) numerals [Tables 8-1, 8-2 39], was also published on 15 June 2009 [25]. The URL that contained links to each of these three files is archived at <http://www.webcitation.org/5hYWAc dhW>.

2.3.2 Control data set: other recent, pre-2009 presidential-election first rounds

For comparison with the Iranian election, data from the first round of voting in several comparable, recent elections were gathered. Some negative and positive criteria for attempting to select statistically and sociologically comparable data sets are the following.

- (1) Parliamentary/congressional elections would differ from the 2009 Iranian presidential election in that in the former case, voters often choose political parties rather than focus on candidates, so the types of voting behaviour patterns may not sufficiently match those in presidential elections.
- (2) Presidential-election second rounds usually only have two candidates, so would restrict the parameter freedom of the random variables more than in a presidential election with many candidates. Single-round, non-preferential elections also have less parameter freedom than runoff or preferential elections, since the former discourage voting for minor candidates.
- (3) Countries with low populations risk having a narrower logarithmic range of voting area sizes (and, hence, a narrower logarithmic range in votes for any given candidate). Also, minority candidates in lowly-populated countries are likely to have votes in some areas that tend towards the zero-vote limit mentioned above.
- (4) Practical (i.e. online) access to the data is preferable so that the data can be independently, publicly archived and independent researchers can obtain the (possibly updated) data directly from the official sources.
- (5) There should be no major controversies regarding suspected artificial interference in the election results.

Table 1 shows the characteristics of online presidential-election first rounds from relatively highly populated countries from different geographical areas that were found for this study, in addition to those of the Iranian 2009 election. These include a well-established democracy in Western Europe, two countries that have had regular elections for a few decades since emerging from left-wing and right-wing authoritarian political systems, and the Iranian 2005 first-round presidential election. The sociological diversity among these countries and the time span of sustained electoral politics and elections among them should imply conservative results, in the sense that an election which is statistically unusual in comparison

¹Retrieval of the archival version may require manual removal of html header text and tab symbols, e.g. using standard GNU/Linux tools, `tail -n 1236 MOI.bin | head -n 1235 | sed -e 's/^\\t\\t//> MOI.xls`, where the retrieved file is named `MOI.bin` and the output file with head and tail removed is to be named `MOI.xls`.

Table 1. Recent presidential-election first rounds, each with n candidates from m voting areas, and a total of $\sum x_i$ votes.

country	date	m	$n(n_0)^a$	$\sum x_i^b$	admin. division	source
France	2002-04-21	100	16	29,183,176	department	[21]
Iran	2005-06-17	326	7	28,155,678	shahrestan	[20]
Poland	2005-10-09	379	11(12)	14,993,138	powiat	[26]
Brazil	2006-10-01	2832	5(7)	94,785,276	“electoral zone” ^c	[35]
France	2007-04-22	100	12	36,674,996	department	[22]
Iran	2009-06-12	366	4	39,245,991	shahrestan	[23]

^aThe analysis is carried out for n candidates, where those candidates among the original n_0 who have 1% or more votes of 1 or 0 are excluded (Section 2.2.2).

^bTotal votes $\sum x_i$ are the sums of total votes per i -th region, x_i , stated in the source, except for the Iranian 2005 election, where x_i are summed from the candidates’ votes in the voting areas. The total votes $\sum x_i$ may differ from the full total due to groups not included in the main geographical regions, e.g. voters living abroad, on ships, etc.

^cThe “electoral zones” in the Superior Electoral Court’s data are smaller than states and larger than municipalities.

to this calibration set of elections is unusual in comparison to a diverse set of sociological situations rather than in comparison to a narrow set of sociological situations. The administrative division levels (called “voting areas” in this paper) were chosen with the aim of obtaining values of m that are logarithmically close to the number of administrative divisions in the Iranian 2009 presidential-election first round. The alternative to choosing the “electoral zone” level in the Brazilian election would have been to choose the state/federal district level, giving $m = 27$. The Iranian 2005 election first round should be that which most closely mimics the statistical characteristics of that of Iran in 2009. Criterion (5) does not necessarily imply that the elections were totally free of artificial interference. However, if some artificial interference did occur in these control elections, then that should most likely lead to an underestimate of the significance of an unusual statistical signal in the Iranian 2009 presidential-election first round. Hence, this approach is conservative.

2.4 Pre-election polls

At least twenty-seven pre-election opinion polls of Iranians’ voting intentions were carried out from March to June 2009 and were publicly known and referenced by the English-speaking community before the day of the election [11]. Among these, six concerned Tehran only, two only giving rankings rather than percentage support, one concerned unpopularity rather than popularity, and one only concerned employees, leaving 17 nationwide polls from 11 journalistic sources.

The sources of these polls are diverse. Several polls are reported by news organisations or from sources that are generally considered to be close to one or more of the candidates. Thus, systematic biases could be suspected to be present either favouring the incumbent candidate or favouring candidates representing either reformist or conservative opposition sectors. The selection process implicit in the creation of this list is a stochastic process that depends on the various positive and negative feedback mechanisms in the editing of the English-language Wikipedia in general, the way that these processes applied to the English-language Wikipedia article concerning the Iranian 2009 presidential election in particular, and the way that English-Persian bilingual editors participated in adding to the English-language

article and correcting errors, possibly improving consistency between the versions of the page in the two languages.

From the article's creation on 2 May 2007 to the last edition on the day (UTC) preceding the election, 11 June 2009, a total of 345 edits were made [12]. The number of individual human authors (excluding robots and grouping together authors identified by nearly identical Internet Protocol numbers) is about 85. From the edit made on 10 April 2009 adding information on an opinion poll up to 11 June 2009, a total of 212 edits were made. Editing activity on the corresponding Persian-language article started later and was more intense. From its creation on 25 August 2008 to the last edit on 11 June 2009 (UTC), a total of 969 edits were made [3]. The number of individual human authors (excluding robots and grouping together authors identified by nearly identical Internet Protocol numbers) is almost identical to that of the English-language article, i.e. about 84. Information on pre-election polls in the Persian-language article is not in the same tabular format as in the English-language article. Consistency between different language versions of a Wikipedia article is decided by consensus among bilingual editors. About seven authors have edited both language versions.

These large numbers of editors and edits on the English and Persian versions of the Wikipedia article, together with the highly open and well-documented nature of the editing process, suggest that the English-language compilation of pre-election opinion polls of the evening of 11 June 2009 is probably less biased than any other compilation available prior to the date of the election. Table 2 lists these polls.¹

2.4.1 Pre-election poll analysis

While the list of pre-election polls given in Table 2 is probably less biased than any other English-language compilation available prior to the date of the election, at least two reasons can be invoked to exclude some of the 17 polls listed in Table 2. The two subsets of polls which could be excluded generally disfavour candidate A and favour candidate M.

Firstly, the sources of seven polls (favouring candidate M since May 2009 and candidate R five days before the election) are stated as being a combination of a field survey and an internet survey, with a warning regarding a likely demographic bias in favour of voters connected to the internet [36, 37]. Moreover, robots can credibly introduce significant systematic errors in an internet survey, but (even as of 2012), are extremely unlikely to introduce any errors in the raw datasets of telephone or door-to-door surveys. Unless additional information estimating the systematic errors in these surveys is added to the analysis, the exclusion of these polls needs to be considered.

Secondly, two sources (favouring candidate M) are no longer online [5, 13]. No online archival copy is known, so their data remain public only indirectly [11]. Hence, exclusion of these two polls should be considered. Alternatively, censorship probabilities could be included in the analysis. Websites more likely to favour candidates disfavoured by the government could be considered to have a higher chance of being closed down by administrative, political or financial methods. In order to derive results that are conservative in the sense of favouring pro-government candidates, these probabilities are neglected in this paper.

Elementary statistical tests are applied to different subsets of the pre-election poll data shown in Table 2, excluding either the partially internet-based polls, the polls that do not have a known online archive, or both, to see if the various subsets

¹Another pre-election poll, carried out by non-Iranian organisations by telephone in early May 2009, became known to the English-language online community *after* the election [38]. Inclusion of this additional poll would not modify the results significantly.

Table 2. Intentions of q_i voters to vote for candidate j with probability w_{ij} in the 2009 Iranian presidential first-round election, in the i -th of 17 pre-election opinion polls [11].

date ^a	q_i	w_{iA}	w_{iR}	w_{iK}	w_{iM}	source	organ. ^b
< 10 June 2009	-	60% ^c	-	-	30% ^c	[4]	-
< 9 June 2009	7900	23 %	-	-	55.5% ^d	[32]	-
< 9 June 2009	1743	25.5 %	30.8 %	6.1 %	37.6 %	[37] ^e	R&B
< 7 June 2009	16,500 ^f	61.1% ^c	-	-	27.2% ^c	[4]	IRIB
< 6 June 2009	16,945	22.5 %	4 %	7.5 %	64 %	[5] ^g	-
< 3 June 2009	-	29.5 %	25.2 %	7.5 %	37.5 %	[37] ^e	R&B
< 1 June 2009	-	53 %	-	-	36 %	[28]	Raja
26 May–5 June 2009 ^h	300,000	24.61 %	10.14 %	10.72 %	54.53 %	[17]	-
31 May 2009	-	32 %	27 %	6 %	36 %	[36] ^e	R&B
< 31 May 2009	77,058	33 %	27 %	3 %	36 %	[36] ^e	Baz.
< 27 May 2009	1650	35 %	-	-	54 %	[13] ^g	-
< 26 May 2009	-	34 %	-	-	38 %	[30]	Ayan.
5 May 2009	-	38.1 %	14.7 %	11.1 %	30.6 %	[37] ^e	R&B
3–4 May 2009	-	59 %	-	-	22 %	[29]	Raja
< 3 May 2009	-	54 %	-	-	22 %	[29]	gov.
4 April 2009	-	40.4 %	0.6 %	7.9 %	23.5 %	[37] ^e	R&B
5 March 2009	-	43.7 %	0 %	7.2 %	12.9 %	[37] ^e	R&B

^aThe dates of polls with upper limits to the polling date are interpreted to be one day before the upper limit implicit in the source.

^bPolling organisation if known: Ayandeh News (Ayan.), Baznevis (Baz.), “government” (gov.), Islamic Republic of Iran Broadcasting (IRIB), Rahbord & Danesh (R&B), Rajanews (Raja).

^cThe source values for Tehran and the countryside are combined in the proportions of the official results, i.e. 4,179,188:35,066,803 [23].

^dThe source states 54–57%; the mean 55.5% is adopted here.

^ePoll includes internet component. The polls of 5 March, 4 April, and 5 May also estimate 36.2%, 27.6%, and 5.3% support, respectively, for candidate Kh, who announced his withdrawal on 16 March.

^fThe source states 16,000–17,000.

^gSource no longer online and no archival copy is known.

^hThe median date (31 May) of the ILNA poll is adopted for that poll.

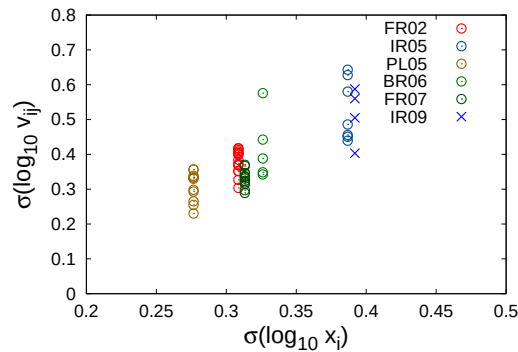


Figure 1. Logarithmic widths (standard deviations) of per-candidate vote distributions v_{ij} vs widths of total vote distributions x_i . From left to right: Poland 2005, France 2002, France 2007, Brazil 2006, Iran 2005 ($\odot$); Iran 2009 ($\times$).

are self-consistent, and if they are consistent with the official results [23]. Moreover, after applying both exclusions, it still remains credible that among the remaining eight polls, either those polls favouring candidate A or those disfavouring A could contain systematic errors favouring or disfavouring A, respectively. Thus, both of these subsets are considered separately, for self-consistency and for consistency with the official results.

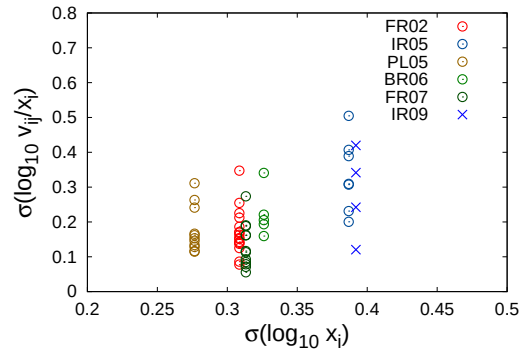


Figure 2. As for Fig. 1, logarithmic widths of per-candidate voting rates $\widetilde{v}_{ij}$ vs widths of total vote distributions x_i .

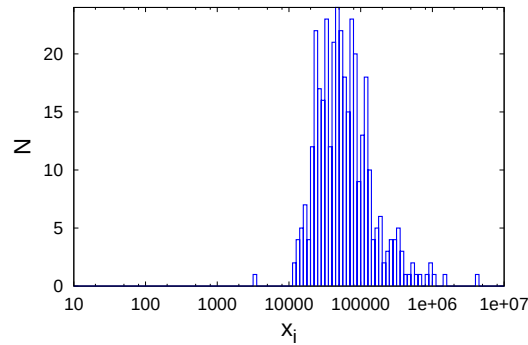


Figure 3. Logarithmic distribution of the total vote counts x_i .

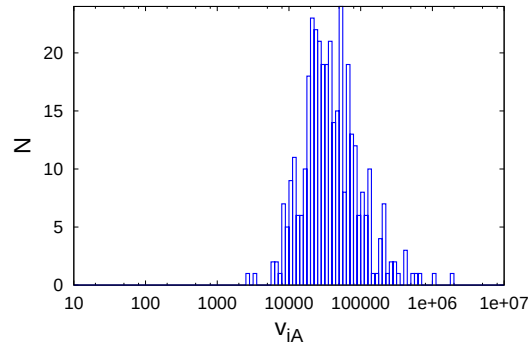
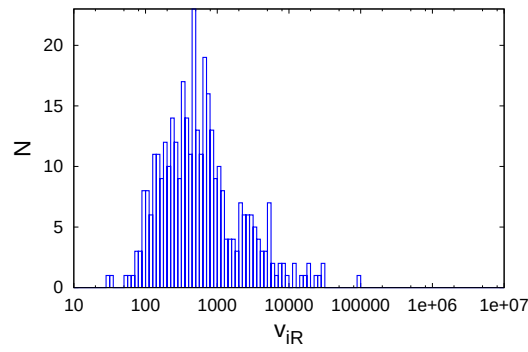
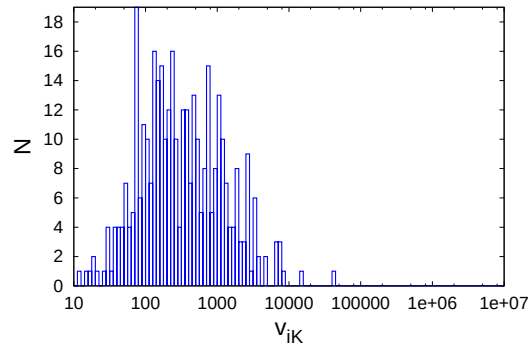
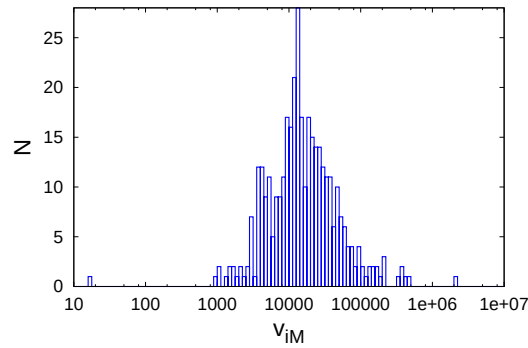


Figure 4. Logarithmic distribution of the vote counts v_{iA} for candidate A.

Can the stated numbers of people interviewed be used to assume that the error in a poll is dominated by the random error implied by the Poisson limit associated with the number of people interviewed, i.e. assuming negligible additional error from the required demographic profile corrections? This would seem unrealistic. For example, the prior-to-7 June IRIB survey [4] and the 26 May–5 June ILNA survey [17], taken during approximately the same period, should differ in their results by less than about 3% (95% confidence) if only the Poisson limit errors are considered. Yet their estimates for candidates A and M differ by 36.5% and 27.3%, respectively. Clearly, the errors in at least one of these two cases are about an order of magnitude larger than the minimum random error attainable for the stated sample sizes. Moreover, nine of the polls do not state their sample size at all.

Figure 5. Logarithmic distribution of v_{iR} , the vote counts for R.Figure 6. Logarithmic distribution of v_{iK} , the vote counts for K.Figure 7. Logarithmic distribution of v_{iM} , the vote counts for M.

Hence, for a conservative analysis of the data, linear least-squares best fits of voting intentions as a function of civil date $t \in [1, 100)$ in days, where $1 = 5$ March 2009 and $100 = 12$ June 2009 are made for the different subsets, giving equal weights to all polls in a given subset. The quality of the fit is estimated using χ^2 of the residuals, where the variance $\text{Var}(w_{ij}) = (3\%)^2$ is assumed. This corresponds to assuming that the error is no more than the random error for a survey of 1000 people obtainable from the Poisson limit. The cumulative probability distribution of the χ^2 distribution then gives an estimate of the degree to which the polls of a given subset agree with each other.

Since the assumed variance should in principle be an overestimate, but in practice may be an underestimate, the standard errors from the fits are then used to estimate the uncertainty in the expected results of the 12 June vote for the majority candidates A and M. The errors are assumed here to be random and normally dis-

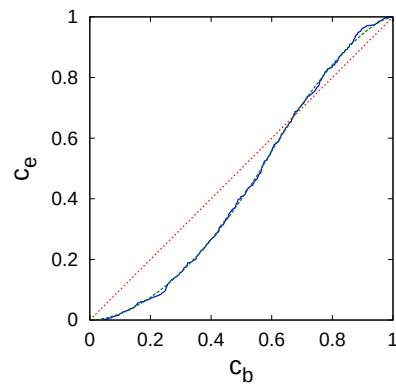


Figure 8. Calibration of confidence levels using the control data set. The empirical confidence levels c_e are shown against local bootstrap confidence levels c_b as a thick curve, and the smooth fit is given by Eqs (8) and (9). If the bootstrap confidence intervals were unbiased and symmetric, then the relation $c_e = c_b$ would hold, as illustrated for comparison.

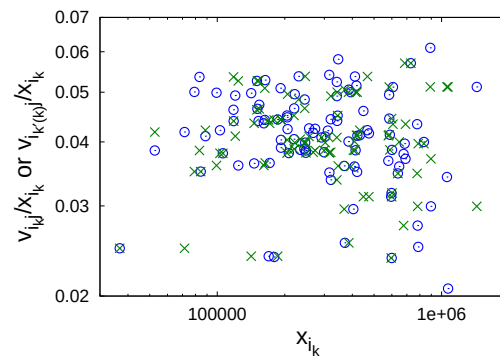


Figure 9. Example of a local bootstrap realisation for candidate 1 in the French 2007 presidential-election first round, showing official voting rates $\widehat{v}_{ij}$ ($\odot$) and simulated voting rates $\widehat{v}_{i_k'j}$ ($\times$).

tributed. This is clearly only a rough approximation. Linear fits to the complement of the major candidates' pre-election support are also considered.

3. Results

Figure 1 shows that the two Iranian presidential elections have a higher dispersion in the total population sizes of voting areas (shahrestans) than the elections used for comparison. Unsurprisingly, the dispersions in the logarithmic vote counts for the individual candidates are also higher in the Iranian elections compared to the other four. Thus, the Iranian elections are more likely to approach the Benford's Law limiting case for first digit distributions than the other elections, since their logarithmic distributions are wider. Moreover, the widths are roughly 0.5, so that, to the degree that each distribution of v_{ij} is log-normal, about 68% of the corresponding cumulative distribution should cover one decade. This is another factor in favour of the Iranian elections approaching the Benford's Law limit.

Figure 2 shows that this is not just an effect of the dispersion in the population sizes of the voting areas. The voting rates $\widehat{v}_{ij}$ in the Iranian elections generally are more widely log-dispersed than in the other four elections. Figure 3 shows the distribution of the total votes per voting area, and Figs 4–7 show the distributions of the votes for the four candidates.

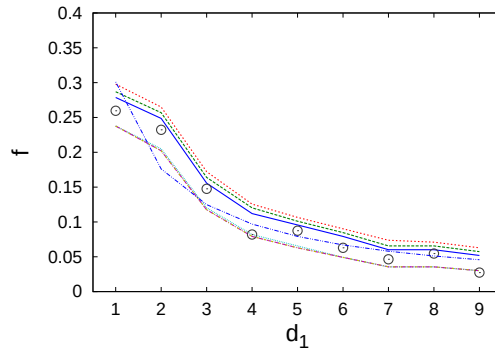


Figure 10. Frequency distribution $f(d_1)$ of the first digit d_1 for candidate A ($\odot$) in the Iranian 2009 presidential-election first round [23]. Lower and upper confidence levels of $c_e = 0.05\%, 0.5\%, 2.5\%, 97.5\%, 99.5\%, 99.95\%$, using the local bootstrap simulations and the empirical calibration given in Eqs (8) and (9), are shown. The lower three confidence levels are close to one another and difficult to distinguish in the plot. The limiting case of the Benford's Law first-digit distribution is shown as a thin line.

3.1 Empirical, local bootstrap model

Figure 9 shows an example local bootstrap, as described in Definition 2.1, for an arbitrarily selected candidate (the first in the table) in an arbitrarily selected election (the French 2007 election first-round). This illustrates how closely the local bootstrap imitates the data set. The logarithmic dispersion of voting rates $\widetilde{v}_{ij}$ for this candidate is greatest in highly populated voting areas, i.e. for $x_i \sim 10^6$. The bootstrap imitates this wide dispersion at $x_i \sim 10^6$ with little obvious effect on low population voting areas. At lower populations, there are some outliers with low voting rates in the real data. The bootstrap realisation appears to reproduce similar outliers, while also matching the bulk of the $\widetilde{v}_{ij}$ distributions in these lower population voting areas. Overall, the realisation appears to closely match the distribution of the official data, possibly exaggerating the numbers of outliers. Subjectively, the figure suggests, as expected, that the simulations should be conservative. They are likely to statistically mimic features of the data even if those features constitute artificial interference in the data set. Only a very unusual interference in the data set is likely to be considered extreme when compared to an ensemble of local bootstrap simulations.

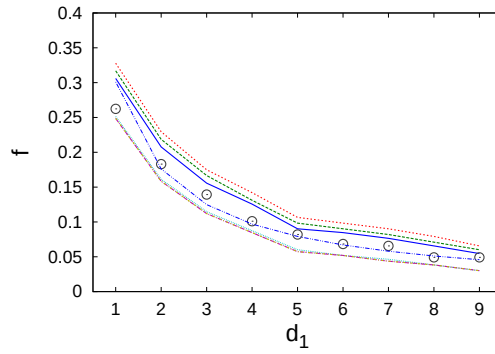
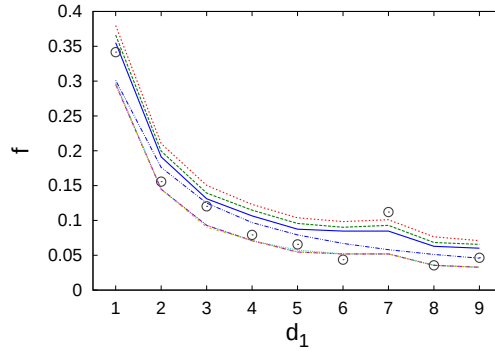
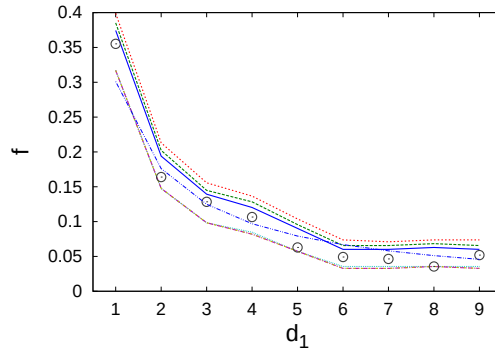
Another characteristic that is clear in the figure is that which is intrinsic to bootstrap methods: a particular value of $\widetilde{v}_{ij}$ may occur several times among the values $\widetilde{v}_{i'j'}$, introducing a discreteness effect. However, in the present method, what is of interest for first digit frequencies is $\widetilde{v}_{i'j'}x_i$. This should remove most of this discreteness effect.

In order to estimate bootstrap confidence levels $c_{b,j}(d)$ for rejecting each of the candidates' first digit frequencies in each of the elections in the control data set, 10^5 local bootstrap simulations were generated. Together with c_e defined by the actual frequency distribution of these values, least-squares fits were found numerically as described above, giving

$$\beta^- = 1.824, \beta^+ = 1.487, c_b^* = 0.566 \quad (9)$$

in Eq. (8).

Figure 8 shows that the local bootstrap confidence intervals are biased and skewed. They are conservative at high confidence levels at both the lower and upper limits. Taken literally, the local bootstrap confidence intervals fail to reject the actual first-digit frequencies as often as they should. For example, none of the 459 first-digit frequencies for candidates in the control elections fall below the lower

Figure 11. First-digit frequency distribution $f(d_1)$ for candidate R, as for Fig. 10.Figure 12. First-digit frequency distribution $f(d_1)$ for candidate K, as for Fig. 10.Figure 13. First-digit frequency distribution $f(d_1)$ for candidate M, as for Fig. 10.

cutoff of the 95% confidence interval, i.e. the 2.5% percentile, as calculated directly by the bootstrap method for each of those candidates, even though in principle about 11 should. The calibration given by Eqs (9) and (8) and illustrated in Fig. 8 should provide more accurate (less conservative) estimates of confidence levels.

The inverse of Eq. (8) can also be useful, i.e.

$$c_b(c_e) = \begin{cases} (2c_e)^{1/\beta^-} c_b^* & , c_b < c_b^* \\ 1 - \left\{ [2(1 - c_e)]^{1/\beta^+} \right\} (1 - c_b^*) & , c_b \geq c_b^* \end{cases} \quad (10)$$

using β^-, β^+ and c_b^* from Eq. (9).

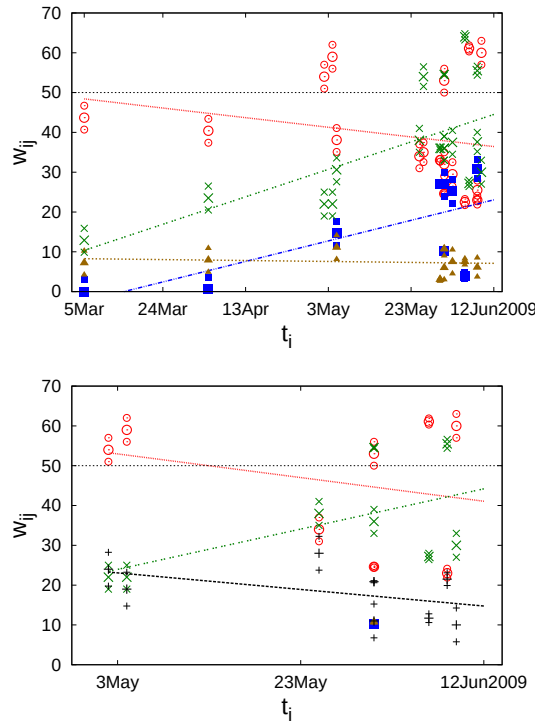


Figure 14. Pre-election poll voting intentions $w_{ij}\%$ (Table 2) for candidates $j = A$ ($\odot$), R (squares), K (triangles), M ($\times$), versus date t_i in days, where $100 = 12$ June 2009 (the election date). Error bars defined as $100\%/\sqrt{q_i}$ are shown as corresponding small/thin symbols, where $q_i := 1000$ if no value is stated in the table. Linear least-squares equal-weighted fits to $w_{ij}(t_i)$ for each j -th candidate are shown, including extrapolations to 12 June. The 50% threshold for winning the election is shown. *Upper panel*: all four candidates, all polls; *lower panel*: candidates A, M and RKO (total support for candidates R, K and “other” inferred from support for A and M) (+), excluding partially internet-based and publicly unarchived polls.

3.2 Iranian 2009 presidential election

Figures 10–13 show the first digit frequencies of the four candidates of the Iranian 2009 first-round presidential election using the corrected confidence intervals c_e corresponding to two-sided confidence levels of 95%, 99% and 99.9%.

It is clear in Fig. 12 that candidate K has an excess frequency of votes v_{ij} that start with the digit 7, to a calibrated significance well above 99.9% (two-sided). The bootstrap estimate is $c_{bK7} > 99.924\%$, and the calibrated estimate is $c_{eK7} > 99.9960\%$, i.e. $1 - c_{eK7} \approx 4 \times 10^{-5}$. Since this could be considered one of 36 independent tests for the full Iranian 2009 data set, a Šidák-Bonferonni correction factor [1] of 36 gives

$$p_7 < 1.5 \times 10^{-3} \quad (11)$$

for rejecting the full data set. Even if the bias/skewness correction were ignored, i.e. if c_b were used, the full data set would still be rejected at a level of $p < 3\%$ based on the excess of first-digit 7 votes for K, i.e. $36(1 - c_{bK7}) \approx 0.027$.

3.3 Pre-election opinion polls

3.3.1 Candidates A and M

Figure 14 shows the pre-election poll data (Section 2.4, Table 2). Both panels include equal-weighted, linear, least-squares fits to the temporal evolution of the

Table 3. Internal consistency (confidence level $p < p_{\chi^2}$) among subsets of the pre-election polls and external consistency (confidence level $p < p_{y_j}$) of official result $y_j / \sum_j y_j$ with linear least-squares equal-weighted best fit to poll evolution $w_{ij}(t_i)$.

selection					p_{χ^2} (internal)					p_{y_j} (external)				
i. ^a	u. ^b	A ^{+c}	A ^{-d}	N_p ^e	A	R	K	M	RKO ^f	A	R	K	M	RKO ^f
Y	Y	Y	Y	17	0	10^{-11}	0.736	0	—	10^{-16}	10^{-11}	10^{-12}	10^{-5}	—
N	Y	Y	Y	10	0	—	—	0	0.064	10^{-4}	—	—	0.003	10^{-4}
Y	N	Y	Y	15	0	10^{-4}	0.638	10^{-15}	—	10^{-11}	0	10^{-8}	10^{-3}	—
N	N	Y	Y	8	0	—	—	0	0.064	0.004	—	—	0.043	10^{-4}
N	N	Y	N	5	0.273	—	—	0.153	0.948	0.131	—	—	0.604	10^{-4}
N	N	N	Y	3	0.447	—	—	0.050	0.706	0	—	—	0.001	10^{-3}

^aAre the partially internet-based polls included (Yes/No)?

^bAre the publicly unarchived polls included?

^cAre polls where $w_{iA} \geq 50\%$ included?

^dAre polls where $w_{iA} < 50\%$ included?

^eNumber of polls in subset.

^fThe sum of votes for R, K, and other (O) are inferred from the votes for A and M in the cases where the partially internet-based polls are excluded. The official result in this case is the sum of votes for R and K.

Table 4. Expected values $E(w_{ij}^*)$ and standard errors $SE(w_{ij}^*)$ of 12 June vote proportions $w_{ij}^* := w_{ij}(t = 12 \text{ June})$ for candidates $j = A$ and M implied by linear least-squares equal-weighted best fits to $w_{ij}(t_i)$.

i. ^a	u. ^b	A ^{+c}	A ^{-d}	$E(w_{iA}^*)$	$SE(w_{iA}^*)$	$E(w_{iK}^*)$	$SE(w_{iK}^*)$	$E(w_{iM}^*)$	$SE(w_{iM}^*)$	$E(w_{iRKO}^*)$ ^{ef}	$SE(w_{iRKO}^*)$ ^f
Y	Y	Y	Y	36.4%	3.2%	7.1%	0.9%	44.5%	2.5%	—	—
N	Y	Y	Y	36.3%	6.5%	—	—	50.1%	5.5%	13.6%	2.4%
Y	N	Y	Y	38.4%	3.5%	7.0%	1.1%	40.8%	2.1%	—	—
N	N	Y	Y	41.1%	7.4%	—	—	44.2%	5.2%	14.7%	2.8%
N	N	Y	N	58.9%	2.3%	—	—	32.3%	2.7%	8.7%	1.2%
N	N	N	Y	18.7%	4.6%	—	—	62.8%	8.9%	18.5%	4.3%

abdcdf As per Table 3.

^eOfficial result [23]: $y_{RK} = 2.5\%$.

candidates' support, which are extrapolated to the election date. Interpreted literally, the extrapolations of the linear fits to the election data would imply that candidates A and M each won about 40% of the votes, i.e. a second-round of the election was necessary. However, it is also clear that this is a best-fit that differs from the individual poll results for A and M by ~ 10 –20% during the fortnight preceding the election. This is the case for both panels, i.e. whether including or excluding the partially internet-based and publicly unarchived polls.

For candidates A and M, the quality of the linear fits in these two panels, assuming $\pm 3\%$ random, normally distributed errors (68% confidence) as stated above (Section 2.4.1), is given for the full set of polls by $\chi^2 \approx 305$ and 194, respectively, for $d = 16$ degrees of freedom, and for the smaller set of polls by $\chi^2 \approx 188$ and 92, respectively, for $d = 7$. This is why the confidence levels, shown in the first and fourth rows of Table 3, respectively, show the χ^2 confidence levels estimated as $p_{\chi^2} \approx 0$.

Table 4 shows the estimates of what the results of the first-round election on 12 June should have been if the linear fits and the residuals to the fits are used.

Even in the fourth row, where the exclusions have been applied and five out of the eight remaining polls are from sources that could be suspected of being more likely to favour candidate A than candidate M [4, 28, 29], the best estimate is that no candidate was expected to obtain over 50% support on 12 June. The precision is about 200–400 times worse than that of the official results, where a Poisson error would be about 0.02–0.03% for candidates A and M, rather than about 5–7% as estimated here. This is partly because of the disagreement between the different polls. Unless an assumption about which subset of polls is the most reliable is made, it is difficult to make preciser estimates. Hence, assuming normal error distributions, the probability of a first-round win by candidate A or M is about $p < 23\%$ or $p < 26\%$, respectively, from the fourth row of Table 4. This is not a significant rejection of a win by either candidate. A larger subset of the polls (higher row in the table) can increase or decrease the probability of a first-round win by M, but in all three cases gives a significant rejection of a first-round win by A. If the official results have been interfered with, then the estimate in the fourth row of Table 4 may be more *accurate* than the official results, despite being a few orders of magnitude less *precise*.

The final two rows of Table 4 show that if either the five polls favouring or the three polls disfavouring candidate A (after the exclusions have been applied) are considered alone, then either candidate A or M is expected to have very significantly or insignificantly won the first-round election, respectively, assuming normal error distributions. The corresponding values of p_{χ^2} in the last two rows of Table 3 show that the self-consistency among the polls in either subset considered alone is quite high. The worst case is $p_{\chi^2}(M) = 0.050$ for polls disfavouring A. Inspection of the lower panel of Fig. 14 suggests that this is because $w_{iM}(t_i)$ has a negative second derivative in this case, i.e. M's support appears to have saturated at a little over 50% about a week before the first-round election, if this subset of polls is considered alone and without the addition of systematic error. These results are unsurprising: polls favouring A are self-consistent and favour A, and polls disfavouring A are marginally self-consistent for a linear fit and favour M.

3.3.2 Candidate K

While conclusions regarding whether candidate A or M or neither won the election first-round are weak due to the disagreement between the pre-election polls, the upper panel in Fig. 14 and the first and third rows of Table 3 show that despite this disagreement, all the polls that published estimates of K's support agree with each other for the best linear fit for a $\pm 3\%$ assumed error. Moreover, the standard error in the fit on the day of the election first-round is small. The first and third rows of Table 4 show that for the two subsets that contain sufficient information on K's vote support in order to make a linear fit to the data, the expected result for K is $w_{iK}^* \approx 7 \pm 1\%$, with a negligible dependence on inclusion/exclusion of the two publicly unarchived polls.

This very strongly disagrees with the official result in the MOI data [23]. Candidate K only got 0.838% support in the official result. This is about six standard errors lower than the mean. This is why $p_{y_K} \ll 1$ in Table 3. The slope of the fit is $(-1 \pm 3) \times 10^{-4}$, i.e. $(-1 \pm 3)\%$ over 100 days, for both poll subsets. That is, K's support is consistent with being constant over three months. Assuming that K's support did not change at all gives only slightly differing results to the above: $w_{iK}^* \approx 7.4 \pm 1\%$ and $p_{y_K} \sim 10^{-15}, 10^{-12}$ for the two poll subsets.

In other words, what the pre-election polls best agree on is that K's support was very close to constant for approximately three months. They agree on this with a margin of error that is reasonable for carefully carried out opinion polls of typical sample size (~ 1000) and consistent with the scatter among the polls' estimates

for K's support. These same polls reject the official result to very high significance.

Nevertheless, most of the polls that reported on support for candidate K are partially internet-based. Hence, several interpretations of this result are possible, including:

- (1) there was a swing against K during the last several days preceding the election, i.e. he lost about 90% of his support during the days preceding the election; or
- (2) the partially internet-based polls overestimated K's support by about a factor of 10; or
- (3) the official result for K was artificially modified.

Case (1) seems unlikely. After three months of stable support, it seems sociologically unrealistic that 90% of K's support would disappear in several days, barring a dramatic, well-mediatised event that motivated an overwhelming majority of previously "committed" supporters of K to suddenly drop their support. The highly significant excess of vote counts with first-digit "7" for candidate K found using the local bootstrap method (Section 3.2) is consistent with case (3). However, without a lot more information on the calibration methods and demographic profiling used in these polls, it would be difficult to exclude case (2).

3.3.3 Candidates R, K, and "other" (O)

Could it be possible to estimate the expected 12 June result for K without relying on the partially internet-based polls, despite the fact that most other polls did not state the support for K? Contingent on some credible assumptions, yes, though indirectly. The complement of the polls' estimates of support for candidates A and M can be used. In principle, this should represent the sum of support for candidates R and K and interviewees that were undecided or gave "other" responses to the interview question. This combined voter component is designated as "RKO" in Fig. 14 and Tables 3 and 4.

In fact, Table 2 shows that in those polls including information for all four candidates, the sum of the four candidates' support, or that of the five candidates in the polls of 5 March, 4 April, and 5 May [where support for candidate Kh, who announced his withdrawal on 16 March, is estimated at 36.2%, 27.6%, and 5.3%, respectively [37]], are $100\% \pm 1\%$, except for the prior-to-6 June poll, where the sum is 98% [5]. Since the fractions of undecided/other voters are rarely this small, especially weeks or months prior to an election, this suggests that the tradition in the reporting of opinion polls in Iran is to normalise away undecided/other responses. If this speculation is correct, in the sense that all the polls listed follow this tradition, then

$$w_{i\text{RKO}} := 1 - w_{i\text{A}} - w_{i\text{M}} \quad (12)$$

should be a good approximation to $w_{i\text{RK}}$, i.e. the sum of the support for the two minor candidates, without the complication of undecided/other voters. However, since this is only a speculation, the designation RKO will be retained initially.

The lower panel of Figure 14 and the RKO columns in Table 3 show that just as the larger subset of polls best agree internally on candidate K's vote support and strongly disagree with the official result for K, the situation for RKO is similar. As the second, fourth and lower rows of Table 3 show, p_{χ^2} is higher for RKO than for A and M in any given poll subset, i.e. the vote support estimates for RKO are more self-consistent among the polls for RKO rather than for A or M. The linear best fits for RKO also strongly reject the official result for RK if the "other" component is neglected. These statements hold no matter which subset in Table 3 is chosen, even

Table 5. Expected values $E(w_{ij}^*)$ and standard errors $SE(w_{ij}^*)$ of 12 June vote proportions, as for Table 4, but for 10,000 numerical realisations, using the more accurate method (leading to smaller standard errors) detailed in Sect. 3.3.4.

i. ^a	u. ^b	A ^{+c}	A ^{-d}	$E(w_{iA}^*)$	$SE(w_{iA}^*)$	$E(w_{iK}^*)$	$SE(w_{iK}^*)$	$E(w_{iM}^*)$	$SE(w_{iM}^*)$	$E(w_{iRKO}^*)$ ^{ef}	$SE(w_{iRKO}^*)$ ^f
Y	Y	Y	Y	36.4%	0.6%	7.1%	0.4%	44.5%	0.6%	—	—
N	Y	Y	Y	36.3%	1.0%	—	—	50.1%	1.0%	13.6%	0.9%
Y	N	Y	Y	38.4%	0.7%	7.0%	0.4%	40.8%	0.7%	—	—
N	N	Y	Y	41.1%	1.2%	—	—	44.2%	1.1%	14.7%	1.0%
N	N	Y	N	58.9%	1.7%	—	—	32.3%	1.2%	8.7%	1.1%
N	N	N	Y	18.7%	2.0%	—	—	62.8%	2.9%	18.5%	2.7%

abcdef As per Table 4.

Table 6. Expected values $E(w_{ij}^*)$ and standard errors $SE(w_{ij}^*)$ of 12 June vote proportions, as for Table 4, but for 10,000 numerical realisations and a quadratic least-squares best fit, as detailed in Sect. 3.3.4.

i. ^a	u. ^b	A ^{+c}	A ^{-d}	$E(w_{iA}^*)$	$SE(w_{iA}^*)$	$E(w_{iK}^*)$	$SE(w_{iK}^*)$	$E(w_{iM}^*)$	$SE(w_{iM}^*)$	$E(w_{iRKO}^*)$ ^{ef}	$SE(w_{iRKO}^*)$ ^f
Y	Y	Y	Y	33.3%	0.9%	5.7%	0.6%	46.1%	0.9%	—	—
N	Y	Y	Y	49.4%	1.8%	—	—	38.6%	1.8%	12.0%	1.6%
Y	N	Y	Y	35.7%	1.0%	5.4%	0.6%	42.0%	0.9%	—	—
N	N	Y	Y	54.0%	2.1%	—	—	36.0%	1.9%	10.0%	1.8%
N	N	Y	N	63.7%	3.3%	—	—	25.0%	2.4%	11.3%	2.0%
N	N	N	Y	N/A ^g							

abcdef As per Table 4.

^gToo few polls in this case.

Table 7. Expected values $E(w_{ij}^*)$ and standard errors $SE(w_{ij}^*)$ of 12 June vote proportions, as for Table 4, but for 10,000 numerical realisations and a cubic least-squares best fit, as detailed in Sect. 3.3.4.

i. ^a	u. ^b	A ^{+c}	A ^{-d}	$E(w_{iA}^*)$	$SE(w_{iA}^*)$	$E(w_{iK}^*)$	$SE(w_{iK}^*)$	$E(w_{iM}^*)$	$SE(w_{iM}^*)$	$E(w_{iRKO}^*)$ ^{ef}	$SE(w_{iRKO}^*)$ ^f
Y	Y	Y	Y	35.5%	1.4%	4.8%	1.0%	43.8%	1.3%	—	—
N	Y	Y	Y	57.8%	3.7%	—	—	28.5%	3.2%	13.7%	2.9%
Y	N	Y	Y	37.7%	1.5%	4.2%	1.1%	41.2%	1.3%	—	—
N	N	Y	Y	54.5%	3.7%	—	—	31.5%	3.1%	14.0%	2.9%
N	N	Y	N	68.2%	4.8%	—	—	24.5%	3.3%	7.3%	3.2%
N	N	N	Y	N/A ^g							

abcdef As per Table 4.

^gToo few polls in this case.

if the subset of five polls favouring A is analysed alone. The rightmost two columns in Table 4 show the expected result w_{iRKO}^* for the different poll subsets. It is clear why the official result $y_{RK} = 2.5\%$ is rejected to high significance. However, these rejections depend on the role of the “other” component. Possible interpretations include:

- (1) some or most of the polls in any subset are not normalised polls, so that w_{iRKO}^* includes a component of $\sim 5\text{--}15\%$ of undecided voters, and most of this category of voters made a decision during the last several days to vote

- for A or M, or did not vote; or
- (2) most or all of the polls are normalised polls, and there was a swing against R and/or K during the last day or two preceding the election, i.e. they together lost about 70–90% of their support during the last several days preceding the election; or
 - (3) the official result for R and/or K was artificially modified.

Hence, similar results are obtained whether the partially internet-based polls are included and K's pre-election support and official result are analysed, or if the partially internet-based polls are excluded, the remaining polls are assumed to be normalised, and the combined vote RKO (equal to RK by assumption) is analysed instead. The possibility of "other" voters in pre-election polls becoming invalid voters in the official poll has little effect on the above three possibilities, since the percentage of invalid votes was officially only about 1%.

3.3.4 Sensitivity to linear assumption

Given the inconsistency of the pre-election poll data, it would be difficult to justify fitting a more complex model of the evolution of voters' intentions. Nevertheless, it would be interesting to see how sensitive the conclusions are to the assumption of a linear relation. Given that fractional voting intentions w_{ij} are bound between zero and unity, and that their sum is unity by definition, it is not trivial to choose the most realistic non-linear function for fitting the data. Linear fits themselves are not bound by the unit interval, as can be seen by the extrapolated negative support for candidate R in Fig. 14 (upper) in early March 2009.

In order to qualitatively examine at least some alternatives to a linear fit, quadratic and cubic least-squares equal-weighted best fits have been carried out, using 10,000 numerical realisations in each case in order to simplify calculation of standard errors. In comparison to the method in Sect. 2.4.1, this procedure makes it easier to make the more accurate, but still conservative, assumption that each survey concerned 1000 valid interviewees, and calculate Poisson errors per candidate, rather than assume a 3% error in all cases. This is still conservative in the sense that all surveys with published numbers claim higher sample sizes. Each realisation generates a simulated set of poll results by starting with each known poll result, calculating the Poisson error for the number of interviewees favouring a given candidate, and offsetting the known result by a number drawn from a normal distribution with this standard deviation. A quadratic or cubic least-squares fit is made to each realisation. An ensemble of realisations (with either quadratic or cubic fits) is used to estimate standard errors. Since this method differs from that used above, and should yield smaller standard errors, a linear fit was also performed with this method, in order to enable direct comparison.

Tables 5–7 show the equivalent expected values and standard errors for the first-round election result when linear, quadratic, and cubic least-squares fits are found numerically this way. The results are not generally sensitive to the degree of the polynomial chosen for fitting. The strongest result from the pre-election poll data, that the expected RKO support is much greater than the official result of $y_{RK} = 2.5\%$, appears to be robust. A few cases are sensitive to the linearity assumption, especially for candidates A and M when subsets of polls are analysed. For example, the fourth row of Tables 6 and 7 shows that a first-round win by candidate A is likely in both the quadratic and cubic cases, provided that both the partially internet-based polls and the publicly unarchived polls are excluded. However, to get close to the official result of $y_A = 62.5\%$, the polls disfavouring candidate A also have to be excluded (fifth row).

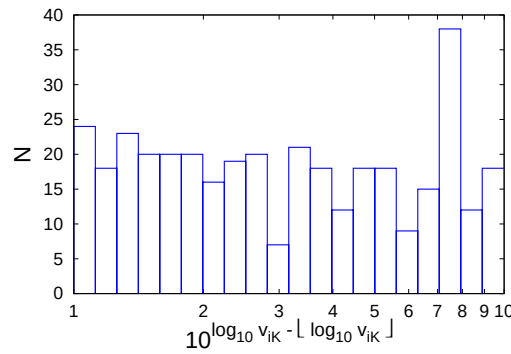


Figure 15. Distribution of the folded logarithmic vote counts for candidate K (cf. Fig. 10, bottom-left).

4. Discussion

The excess of first-digit 7 votes for candidate K seen in Fig. 12 implies a rejection of the null hypothesis, i.e. the hypothesis that no artificial interference occurred in the first-round Iranian 2009 presidential election results, at a confidence level for the full data set of $p_7 < 1.5 \times 10^{-3}$ [Eq. (11)]. This estimate is obtained by a highly conservative method based on the data themselves—a local bootstrap method—and calibrated using similar, prior elections. In other words, it is very difficult to explain this excess of “K7” votes by assuming that among voting areas of approximately the same voting population size, the actual votes for a candidate can be modelled as being randomly drawn from the statistical pattern (distribution) of voting rates for voting areas of about that size for that candidate. Patterns as unusual as the excess of 7-something votes for K did not occur in the five previous presidential elections studied in the present work.

Are there caveats that could weaken this result? What alternative possible explanations, including artificial interference, may there be for this excess?

4.1 Small numbers of data values

The actual number of voting areas whose vote counts for K start with the digit 7 (“K7”’s) is 41. The Benford’s Law limiting expectation for this digit, for 366 voting areas, is 21.2, and the upper confidence limits shown in Fig. 12 lie in between these two values. Intuitively, it may seem difficult to believe that just a few dozen or so voting areas with 7-something votes are unusual enough to reject the credibility of the whole data set. The intuitive error here is probably related to not thinking through the process that (assuming no interference in the data) generated the data. Large numbers of Iranians thought individually, communicated with each other in networks and groups, and eventually made some marks on paper indicating their voting decisions. If Iranian voters in voting areas of about the same size, voting for a given candidate, can be validly modelled by a statistical distribution of a shape determined by the official election results, then there is not much numerical freedom for the vote numbers to vary other than according to that distribution.

On the contrary, the smallness of the number of unusual votes could be considered a factor in favour of an alternative hypothesis of artificial interference, if artificial interference is to be made in a way that is perceived to be unlikely to be detected. Someone wishing to interfere in the data might underestimate the natural constraints for self-consistency within the data set. That is, s/he might intuitively, but incorrectly, guess that modifying a dozen or so values in a table of over a thousand values is statistically insignificant, and, thus, undetectable.

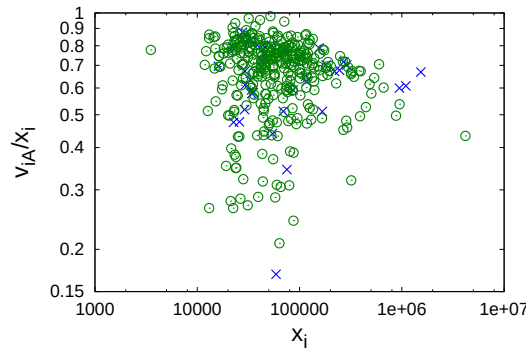


Figure 16. Proportion of vote counts v_{iA}/x_i against total votes v_j , for candidate A. Voting areas are selected by whether K's vote count has the first digit 7 ($\times$) or another digit ($\odot$). The vertical (logarithmic) scales differ greatly between this figure and Figs 17–19, in order to show the structures of the distributions. To compare the widths of the $\log_{10} v_{ij}/x_i$ distributions in the four figures, see Fig. 2.

4.2 Copying error

Could the excess of K7's just be a copying error by employees under pressure in a stressful situation? Various sources of unintentional (but artificial) errors are possible. The present analysis only concerns the data as published by the MOI, not a version of the data closer to the “raw” values provided by individual voting centres. However, this number remained stable from the first to third versions of the data. The number of entries that start with 7 under candidate K in the Arabic-Indic numerals PDF file is 41 [25], in agreement with that of the first XLS file [23].

4.3 Could K's vote distribution be especially spiky/noisy?

Logarithmic intervals of first digits give unequal weights to the different digits if an intrinsic distribution is logarithmically uniform. Could K's vote distribution be in general spiky (noisy), for a reason unrelated to human perception of the decimal system, with some spikes hidden in the low digits (1, 2) because they correspond to wider intervals? Figure 15 suggests that this is not the case. The excess of vote counts starting with 7 is the only obvious spike in K's vote distribution for bins of a size and offset that match the first-digit 7 bin.

Moreover, another property of K's vote distribution suggests that this should be *more* smooth than typical vote distributions. The frequency distribution of votes v_{ij} is a convolution of the distributions of v_{ij}/x_i together with the distribution of effective population sizes x_i of the voting areas. The narrower the distributions v_{ij}/x_i are, the less the convolution will smooth out the underlying distribution of x_i , and in turn, the less the first-digit frequency will be smooth.

In Fig. 2, the second highest value of $\sigma[\log_{10}(v_{ij}/x_i)]$ of any of the presidential elections is that of candidate K. The four widths $\sigma[\log_{10}(v_{ij}/x_i)]$ of the votes for A, R, K, and M are 0.12, 0.34, 0.42, and 0.24, respectively. Candidate K is the candidate in the first-round 2009 Iranian election who should *least* be expected to have a noisy first-digit frequency distribution. The 68% width (assuming normality) of the logarithmic v_{iK}/x_i distribution is approximately an order of magnitude. So unless the shape of the v_{iK}/x_i distribution is exceptionally spiky and correlated between different voting population sizes x_i , the resulting first-digit frequency should be very well smoothed out. The narrowest width is for candidate A. Figures 10–13 do suggest that apart from the excess of first-digit 7 votes (and slight lack of first-digit 6's and 8's) for K, A's first-digit frequency distribution is the most spiky of the four candidates' distributions, even when the local bootstrap method is used to estimate confidence levels.

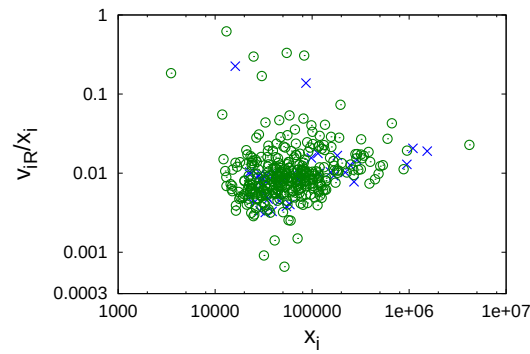


Figure 17. R's proportions of vote counts, as for Fig. 16.

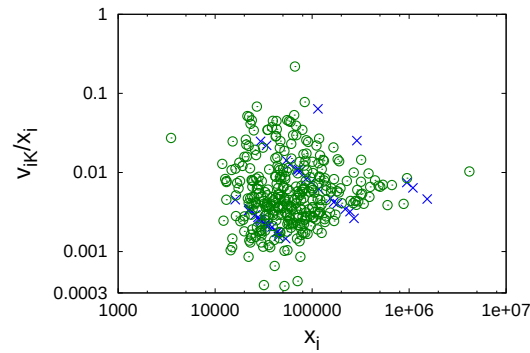


Figure 18. K's proportions of vote counts, as for Fig. 16. The selection by the first digit 7 is clear.

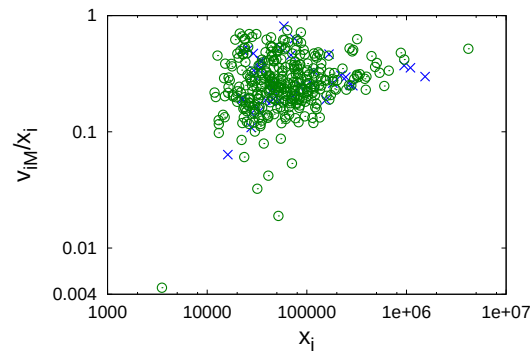


Figure 19. M's proportions of vote counts, as for Fig. 16.

Hence, spikiness in K's vote distribution seems to occur uniquely in the first-digit 7 bin, and it does so in a way that manages to avoid a smoothing effect that is stronger than for the other candidates in the same election and nearly all candidates in the five control elections.

4.4 Does an interference hypothesis lead to any further unusual properties?

4.4.1 The six most populous voting areas and the excess $7d_2d_3d_4$ vote counts for K

If the excess 7's for K indicate interference in the data, then other signs of interference could be expected. A reasonable hypothesis would be that vote counts for one of the major candidates were decreased or increased. Figures 16, 17, 18 and

Table 8. Votes for K and proportion of votes for A for the six voting areas with the greatest numbers of total votes.

voting area	x_i	v_{iK}	v_{iA}/x_i
Tabriz	876919	3513	0.497
Shiraz	947168	7078	0.600
Karaj	950243	8057	0.537
Isfahan	1095399	7002	0.609
Mashhad	1536106	7098	0.669
Tehran	4179188	43073	0.433

19 show proportions of votes that each candidate received as a function of total votes, where those voting areas selected by having 7 as the first digit in K's vote count are highlighted.

It is clear in Fig. 16 that for candidate A, among the six voting areas with the greatest numbers of total votes, the three of these that voted for A in the highest proportions are all selected by the K first digit 7. Data for these six voting areas are listed in Table 8. Fig. 19 shows correspondingly that the three of the six most populous voting areas who voted least for M are also those selected by the K first digit 7. Since the vote fractions for K and R are (in the MOI data) only about 1% each, the high proportions of votes for A necessarily imply low proportions of votes for M.

Are these two subsets of the set of the six most populous voting areas, the $d_1(K) = 7$ group voting over 60% for A, versus the $d_1(K) \neq 7$ group voting less than 55% for A, significantly distinct? The number of points is very small, but the separation between the two populations seems to be clear. The Kolmogorov-Smirnov (KS) test is a non-parametric test that enables the comparison of these two populations. Among the six voting areas with the highest vote numbers, the probability that the three vote proportions for A $\{v_{iA}/x_i \mid d_1(K) = 7\}$ and the three vote proportions for A $\{v_{iA}/x_i \mid d_1(K) \neq 7\}$ are sampled from the same probability density function, or equivalently, that the vote proportions $\{v_{iA}/x_i \mid v_{iA}/x_i > 0.60\}$ and the vote proportions $\{v_{iA}/x_i \mid v_{iA}/x_i < 0.55\}$ are sampled from the same probability density function, is

$$p_{6\text{biggest}}^{\text{KS}} \approx 0.100. \quad (13)$$

This high probability may seem counterintuitive, since the two sets of three values are completely non-overlapping. However, the number of values is extremely small. Non-parametric tests in general have less statistical power than parametric tests, so it is unsurprising that the test is weak.

A stronger test can be applied if it is assumed that the two sub-samples can be approximated as being sampled from normal or log-normal distributions. In this case, the difference in the two means is about 3.7 or 3.5 standard errors in the mean, respectively, i.e. the probability that the two subsamples distinguished by the K excess 7's among the six biggest shahrestans have the same mean is rejected at

$$p_{6\text{biggest}}^{\text{G}} < 5 \times 10^{-4}. \quad (14)$$

The introduction of an assumption regarding possible distribution shapes enables a much more significant rejection of the two samples being drawn from a single

Table 9. Frequencies N of the two-digit votes for K starting with the digit 7.

v_{iK}	70	71	72	73	74	75	76	77	78	79
N	1	2	1	4	1	4	1	2	1	3

distribution than the non-parametric approach. The dependence on the distribution shape (normal vs log-normal) does not appear to be strong.

Could it be expected that, in the absence of artificial interference, there is any statistical dependence between the initial first digit test leading to the excess of 7's for K and the separation of the largest six voting areas into two distinct distributions using this same characteristic? This seems unlikely.

Another coincidence is obvious among the three $7d_2d_3d_4$ vote counts shown in Table 8: the second digit d_2 is zero in all three cases. The standard form of Benford's Law should provide a reasonable approximation to the expected second digit distribution, especially given that K's vote distribution is logarithmically wide, with $\sigma[\log_{10}(v_{ij}/x_i)] = 0.42$. Benford's Law for the second digits gives the probability of 0 as a second digit to be 11.97%, i.e. slightly greater than 10%. (The expected frequency of second digit d is $\sum_{k=1}^9 \log_{10}[1 + 1/(10k + d)]$.) The probability that all three digits are identical—but not necessarily 0—is

$$p_{d_2=d_3=d_4} \approx 0.01037, \quad (15)$$

i.e. slightly greater than $10(0.1)^3$, which would be estimated assuming a uniform linear distribution of values.

Again, this should be independent of the previous probability estimates. There is no reason why dividing the six most populous voting areas into two groups based on their first digit for K's votes being 7, or being in the upper or lower half of voting proportions for A, should have an effect on the second digits of K's votes.

If we suppose that (i) these three voting areas (Shiraz, Isfahan and Mashhad for 7078, 7002 and 7098 votes for K respectively) should have proportions of about 50% for A in agreement with Tabriz and Karaj, which follow an approximately linear upper boundary to A's proportions of votes in the log-log plot in Fig. 16, and if (ii) the total number of votes should remain constant, then from Table 8 this would imply that the correct number of votes for A would be about 473,000 less than in the MOI table. To keep the total number of votes constant, M's, K's and R's votes would also have to be corrected. If these are corrected in proportion to the three candidates' overall vote percentages, then the difference between A's and M's total vote counts would be reduced by about one million votes.

4.4.2 Excess $7d_2$ vote counts for K

The $v_{iK} = 7d_2d_3d_4$ voting areas comprise just a small fraction of the total number of excess first digit 7 votes for K. In Fig. 6, a peak in the 70's, i.e. $7d_2$ votes appears to be strong. Table 9 shows these vote counts.

This distribution is quite literally odd. Most (15 out of 20) of the votes are odd, and the few even votes that occur are themselves distributed with perfect uniformity. Every even number occurs exactly once. The latter necessarily implies the former—these two coincidences are dependent on one another. Given that 20 randomly chosen integers lie in the range from 70 to 79, what is the probability for each even number to occur exactly once (and by implication, for there to be a large majority of odd numbers)? Alternatively, a more conservative estimate of how unusual this distribution is would be to calculate the probability for at least

Table 10. Parity frequencies N of the K7-selected vote counts and the cumulative binomial tail probabilities p .

selection	candidate j			
	A	R	K	M
$N(v_{ij} \text{ even} \mid d_1(v_{iK}) = 7)$	18	15	18	16
$N(v_{ij} \text{ odd} \mid d_1(v_{iK}) = 7)$	23	26	23	25
p	0.266	0.059	0.266	0.106

Table 11. Numbers of polling stations N^{poll} and polling station observers per candidate N_j^{poll} [10]^a and an advisor for M [18]^b; “-” indicates no value claimed.

selection	N^{poll}				ref
	A	R	K	M	
45692	33058	5421	13506	40676	^a
—	—	—	—	25000	^b

15 out of the 20 integers in the range 70 to 79 to be all odd or all even. In both cases, the distribution among the 10 numbers should be logarithmically uniform.

Numerical generation of 20-tuplets of numbers

$$\{\lfloor 10^{\log_{10} 70 + x_k \log_{10}(8/7)} \rfloor\}_{k=1,20}, \quad (16)$$

where x_k is selected uniformly on $[0, 1)$, gives a sample of a logarithmically uniform distribution of numbers in the interval $[70, 80)$. This gives the probability that each even number occurs exactly once to be $p \approx 5 \times 10^{-4}$. The more conservative probability, i.e. the probability that either at least 15 of the twenty $7d_2$ vote counts for K are odd or at least 15 of the twenty $7d_2$ vote counts for K are even, given that they are randomly selected from a logarithmically uniform distribution in the range $[70, 80)$, is (unsurprisingly) higher,

$$p_{7d_2} \approx 0.04, \quad (17)$$

which is identical to the corresponding binomial probability to this precision. Taken alone, this gives the odd dominance of $7d_2$ votes for K to be only marginally significant.

However, Table 10 shows that the vote counts for the full set of 41 K7-selected shahrestans are dominated by odd votes for *all four candidates*. If the parity of the vote counts is an independent statistic, then the overall probability is

$$p_{\text{odd}}^{\text{K7}} \approx 0.00044, \quad (18)$$

so the overall dominance of odd vote counts in K7-selected voting areas *is* highly significant.

4.5 Polling station observer crosschecks

Brill [10] states that all four candidates had thousands of observers at individual polling stations (Table 11). He argues that since there were no disputes about

discrepancies between observers' records of individual polling station results and the Ministry's statement of the same numbers, there cannot be shahrestan-level interference in the vote counts, since the latter must be the sums of the individual polling station results, which were undisputed. Flaws that weaken this reasoning include:

- (1) The K7 anomaly firstly concerns candidate K. Registered observers for K were absent from the vast majority of polling stations, even with only one observer per polling station (Table 11).
- (2) There was sufficient time for top-down artificial interference as hypothesised by [33]: large-scale results were released first, and the detailed results needed only minimal interference in order for the arithmetic to be consistent. Tens of thousands of numbers can be summed in much less than a second on a 2009 personal computer.
- (3) The number of M's observers is disputed between M and governmental authorities (Table 11).
- (4) Given the prepoll analysis here (Sect. 3.3) and Mebane [19]'s analysis, a credible hypothesis would be of major reductions in votes for R and K, major (~ 5 –15%) additions of votes for A, and possibly minor reduction in votes for M. R and K had very few observers, so the chance of their observers discovering polling station discrepancies is low. Candidate A had many observers, but expecting them to complain about the Ministry overestimating the vote counts for A compared to what they observed is unrealistic. People responsible for vote-count alterations were likely to have been aware of the relatively large number of M observers, and may have been aware of their localisation, in order to know at which polling stations interference would have been detectable. Even if the claim that M observers were present at 40,676 of the 45,692 polling stations is accepted, interference in the other 5,016 polling stations may have been sufficient to match the earlier province and shahrestan level results.

Nevertheless, Brill [10]'s argument points to a method of reducing the chance of interference in future Iranian presidential elections. Polling station crosschecks by observers for all candidates at all polling stations, along with allowing mobile and landline telephone networks [18] and the internet to remain active prior to, on and following the election day, would enable the rapid, progressive release of the complete per-polling-station and summed results by all candidates independently. Discrepancies could then be traced and resolved individually.

4.6 *Post-election opinion polls*

Several opinion polls were carried out after the election first round and revealed publicly more than six months later [15]. Given the intense social conflict that occurred following the election, a linear model of public support for the four candidates across both the pre- and post-election periods is likely to be a considerably less realistic model than that presented in the opinion poll analysis in this work, which excludes the post-poll period. These post-election polls find that shortly after and several months after the election first round, candidate A had about 55–65% support.

Accepting the post-election polls as accurate requires accepting the sociological assumption that Iranians' opinions are accurately described by statistical modelling. If Iranians' voting preferences can be modelled as random variables that depend mainly on demographic profiles and are measurable using standard meth-

ods of random sampling, then it is hard to reject the much more conservative local bootstrap model, which uses the official vote counts *as their own model*. In other words, accepting the post-election polls as statistically meaningful requires accepting the official first-round election results to be statistically anomalous. Statistical techniques cannot be accepted in the case with a sample size of $N \sim 10^3$ and rejected where $N \sim 10^{7.5}$ and fewer assumptions are required. Thus, the statistical evidence for artificial interference in the election results remains overwhelming, but the true results remain uncertain. One possible interpretation of the post-election polls is that they could be seen as strengthening the case that if no interference in the vote counts had occurred, then candidate A would have won a first-round absolute majority (row 5 of Tables 3 and 4, excluding partially internet-based polls, publicly unarchived polls, and polls disfavouring A) and it would have been widely accepted as legitimate. For opinion poll data to be more useful in the future, it would be best if a variety of Iranian institutes, well-reputed for independence and statistical methodology, carried out the polls and published detailed reports of methodology and results.

5. Summary and conclusion

A local bootstrap method was defined (Definition 2.1) in order to conservatively analyse vote-count first-digit frequencies in presidential-election first rounds without assuming the Benford's Law limit. Its validity is not necessarily restricted to this domain, but the interest here is that the method can be applied to the 2009 Iranian presidential-election first round. The method was calibrated on a control set of five presidential-election first rounds (2002–2006) and applied to the vote counts per voting area published on 2009-06-14 by the Ministry of the Interior of the Islamic Republic of Iran [23].

The most deviant first-digit frequency is the excess of first-digit 7 votes ($d_1(v_{iK}) = 7$) for candidate K. This is obvious in Fig. 12 and is also present for uniformly spaced bins in the folded logarithmic vote-count distribution in Fig. 15. The uncorrected bootstrap estimate of the significance of the excess implied by the control elections is $1 - c_{bK7} < 8 \times 10^{-4}$. The correction $c_e(c_b)$ for bias and skewness derived from the control elections is given in Eqs (8) and (9). The corrected significance is $1 - c_{eK7} \approx 4 \times 10^{-5}$. A Šidàk-Bonferonni correction factor [1] of 36 gives the probability of finding a first-digit deviation this strong in the full dataset of $p_7 < 1.5 \times 10^{-3}$ [Eq. (11)].

This excess is all the more difficult to explain by any natural voting process given that the logarithmic width (standard deviation) of candidate K's voting rate distribution $\sigma[\log_{10}(v_{iK}/x_i)] = 0.42$ is the second greatest of that of any candidate among the Iranian 2009 and five earlier, similar datasets. This width indicates a wide scatter in people's voting rates for any given voting region size, which should smooth out the resulting vote counts v_{iK} and push first-digit frequencies towards the Benford's Law limit.

If the K7 anomaly is a genuine coincidence, i.e. if a chance numerical coincidence that happens about once in a thousand times to statistically equivalent datasets occurred rather than artificial interference in the data, then selecting voting areas by this criterion should constitute an arbitrary method of selecting a subset of the full data, and should not lead to further unusual events. However, K7 selection *does* lead to several coincidences, one of which appears to favour candidate A, who officially won the election first round, in the six most populous voting areas.

- (1) Of the six voting areas with the greatest total numbers of voters, listed in

Table 8 (see also Fig. 16), three of these (Shiraz, Isfahan, Mashhad) satisfy this criterion, i.e. they have vote totals for K that start with 7 (7078, 7002, 7098 votes respectively). All three of these have greater proportions of votes for A than the other three voting areas. The probability for the two sub-groups of the six big cities to be drawn from the same distribution is mildly rejected by a non-parametric (Kolmogorov-Smirnov) test, with $p_{6\text{biggest}}^{\text{KS}} \approx 0.100$ [Eq. (13)], but strongly rejected by a parametric (difference in means, assuming either normal or log-normal distributions) test, with $p_{6\text{biggest}}^{\text{G}} < 5 \times 10^{-4}$ [Eq. (14)].

- (2) The other surprising property of the three big cities that have total votes starting with the digit 7 for K is that they all have the same second digit. The probability for the second digit of all three to be equal (not necessarily zero) is $p_{d_2=d_3=d_4} \approx 0.01$ [Eq. (15)].
- (3) The voting areas that voted for K with a total of $7d_2$ votes, for any digit d_2 , show an unusual characteristic: 15 of the 20 values are odd numbers, and the even numbers occur exactly once each (Table 9). The chance of the former is $p_{7d_2} \approx 0.04$ [Eq. (17)]. However, a related coincidence is that the full set of K7-selected shahrestans is dominated by odd vote counts for all four candidates, giving $p_{\text{odd}}^{\text{K7}} \approx 0.00044$ [Eq. (18)].

This sequence of significantly unusual properties of the MOI dataset is difficult to explain other than by artificial intervention in the data. While it is true that all of these tests (except for the plan to use a Benford's-Law-like analysis) are *post-hoc* tests, i.e. they were chosen after seeing the data, a Šidák-Bonferroni correction [1] cannot help much. Combining the initial local-bootstrap first-digit test, (1) the six biggest cities parametric test, and (3) the odd dominance of the K7-selected shahrestan vote counts, and a Šidák-Bonferroni correction of C gives

$$p_{\text{all}} < Cp_7 p_{6\text{biggest}}^{\text{G}} p_{\text{odd}}^{\text{K7}} = 3 \times 10^{-10}C. \quad (19)$$

For p_{all} to be rejected with less than 95% confidence, C would have to be greater than 10^8 . It is difficult to imagine that the number of similar statistical tests to those used above is this large. A stronger objection to this estimate of p_{all} is that these different tests may not be statistically independent. However, it is difficult to imagine any natural dependence between these three tests, which each give $p \sim 10^{-3}$ or lower *individually*, that could bring the probability of their combined occurrence to anywhere near unity.

While the first-digit analysis and subsequent unusual characteristics of the MOI data are difficult to explain other than as an effect of artificial interference, these (mostly) do not lead to any indications of what the unaltered data would look like, apart from the six-big-cities K7 selection, in which K7-selected vote counts for A can be interpreted as a misestimate of the true vote, suggesting that the difference between A's and M's vote totals is overestimated by about one million votes.

The set of pre-election opinion polls presented and analysed above in Table 2, Fig. 14, and Tables 3 and 4 gives qualitatively similar results. This list of polls appears to be that which was constructed by what is likely to be the least biased (not necessarily unbiased) method available to the English-speaking community. The Wikipedia article in which the list was originally compiled included 345 edits to the English-language Wikipedia article up to 11 June 2009 (inclusive) [12], 969 edits to the corresponding Persian-language Wikipedia article [3] up to the same date, each by about 84 individual human authors, seven of whom edited both articles, and the number of readers who found the information sufficiently accurate

and reflecting a neutral point of view that they declined to edit the articles is certainly much larger [11, 12]. Linear least-squares fits to the poll set show the following.

- (1) No matter which subset of polls is chosen, the linear least-squares fits to the polls internally agree best on the vote intention evolution either for candidate K alone or for the implied RKO (R plus K plus “other”) vote intentions compared to either of the major candidates, A and M (or R alone in the former case) (p_{χ^2} , Table 3).
- (2) The official results are rejected significantly or to very high significance for nearly every candidate (or RKO) in nearly every subset of polls (p_{y_j} in Table 3, and Table 4).
- (3) Even the most conservatively selected subset of polls, that excludes partially internet-based and publicly unarchived polls and selects those remaining polls that give an absolute majority to candidate A (fifth row of the two tables), rejects the official combined vote for candidates R and K with $p \approx 10^{-4}$, provided that most or all of the polls are normalised (undecided/other responses excluded), which appears to be the case from the available data.
- (4) The poll subset favouring A (fifth row of the two tables) implies a highly significant win for A, while the subset disfavouring A (sixth row of the two tables) implies a weakly significant win for M.
- (5) Unless the subset favouring or disfavouring A is chosen, the best estimate of the election result is that both A and M received less than or just slightly over 50% of the votes ($\sim 35\text{--}40 \pm 7\%$ and $\sim 40\text{--}50 \pm 5\%$, respectively). Neither a first-round win by A, a first-round win by M, or a need for a second round election are significantly excluded.
- (6) Unless the subset favouring or disfavouring A is chosen, the most likely correct result was that neither candidate won the election first round.

Several possible interpretations of (1)–(3) are listed in Section 3.3.3. If the partially internet-based polls overestimated K’s support by about a factor of 10 and if most of the polls that do not state support levels for K are not normalised, then the polls would be consistent with the official results. However, those polls that state support for all the candidates are normalised.

If there was a swing of about 90% against K or about 70–90% against R and/or K during the last several days of the election, i.e. if *nearly all* of their supporters suddenly changed their minds just before the election, then this also would be consistent with the official results. However, the data that are available for K show support consistent with being constant over three months, suggesting that a 70–90% loss of support would require an extremely sudden, strong discouragement of previously loyal supporters.

The most consistent way to explain these results would appear to be the hypothesis of artificial interference in the official results. In this case, if the best estimates for the correct results are obtained from the pre-election polls alone, then this would give about 7% support to candidate K or about 10–20% support to candidates R and K together in the 12 June first-round results (Table 4), in contrast with the official result of 2.5% for the combined vote for R and K. This would be consistent with the first-digit analysis and subsequent unusual properties of the official results, which point directly to extremely unusual properties starting from the excess of vote counts for K that start with the digit 7.

Further evidence for or against either the hypothesis of no interference in the data or the opposite could potentially be obtained by examining the vote counts in the six largest shahrestans (Table 8), and in odd-dominated K7-selected and

shahrestans (Table 10). The voting areas' names are listed in the table published by the MOI [23–25].

Although the local bootstrap method presented here has led to the detection of highly significant anomalies in an electoral poll, *the reverse would not be true in general*. Benford's-Law-like tests may detect anomalies in a dataset but cannot guarantee the absence of anomalies, because many randomising effects can combine to hide artefacts. The anomalies detected in the analysis presented above, where the big city effect suggests an error of about one million votes, may not constitute the full set of anomalies, nor do they provide an estimate of the unaltered data. The compilation of pre-election poll data, despite relatively poor reporting quality in individual sources available directly or indirectly in the English language, probably together provide the most robust estimates available of the unaltered data. Higher quality poll reporting, in particular including data for all candidates and undecided/other percentages, as well as detailed methodologies, would reduce the ambiguities remaining in analyses of pre-election data.

Regarding the Iranian 2009 presidential-election first round in particular, at least three statistical analyses independent of this work have been carried out [6, 9, 19]. Based on very different methods of analysis and to some degree on different data (e.g. including the 2005 election results and excluding pre-election opinion polls), the analyses of Mebane [19] and Chatham House [9] come to conclusions that are compatible with the present results, in particular regarding the unusual nature of the official results for candidate K.

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