Modified Bosonic Gas Trapped in a Generic 3–dim Power Law Potential

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We analyze the consequences caused by an anomalous single-particle dispersion relation suggested in several quantum-gravity models, upon the thermodynamics of a Bose–Einstein condensate trapped in a generic 3-dimensional power-law potential. We prove that the condensation temperature is shifted as a consequence of such deformation and show that this fact could be used to provide bounds on the deformation parameters. Additionally, we show that the shift in the condensation temperature, described as a non-trivial function of the number of particles and the trap parameters, could be used as a criterion to analyze the effects caused by a deformed dispersion relation in weakly interacting systems and also in finite size systems.

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I. INTRODUCTION

The possibility of a deformation in the dispersion relation of microscopic particles, appears in connection with the quest for a quantum theory of gravity [1–9]. This entails, in some schemes, that a possible spacetime quantization has as a consequence a modification of the classical–spacetime dispersion relation between energy E and (modulus of) momentum p of a microscopic particle with mass m [2, 3, 5]. A deformed dispersion relation emerges as an adequate tool in the search for phenomenological consequences caused by this type of quantum gravity models. Nevertheless, the principal difficulty in the search of quantum gravity manifestations in our low energy world, is the smallness in the predicted effects [3, 4]. If this kind of deformations is characterized, for instance, by some Planck scale, then the quantum gravity effects become very small [2, 5]. In the non–relativistic limit, the deformed dispersion relation can be expressed as follows [5, 6]

$$E \simeq m + \frac{p^2}{2m} + \frac{1}{2M_p}\Big(\xi_1 mp + \xi_2 p^2 + \xi_3 \frac{p^3}{m}\Big), \quad (1)$$

in units where the speed of light $c = 1$, with $M_p \simeq 1.2 \times 10^{28} eV$ the Planck mass. The three parameters ξ_1, ξ_2, and ξ_3, are model dependent [2, 5], and should take positive or negative values close to 1. There is some evidence within the formalism of Loop quantum gravity [5–8] that indicates non–zero values for the three parameters, ξ_1, ξ_2, ξ_3, and particulary that produces a linear–momentum term in the non–relativistic limit [7, 9]. Unfortunately, as it is usual in quantum gravity phenomenology, the possible bounds associated with the deformation parameters, open a wide range of possible magnitudes, which is translated to a significant challenge.

On the other hand, the use of Bose–Einstein condensates, as a possible tool in the search of quantum–gravity

manifestations (for instance, in the context of Lorentz violation or to provide phenomenological constrains on Planck–scale physics) has produced an enormous amount of interesting publications [10–19]. It turns out to be rather exciting to look for the effects in the thermodynamic properties associated with Bose–Einstein condensates caused by the quantum structure of space–time.

In a previous report [15], we were able to prove, that the condensation temperature of the ideal bosonic gas, is corrected as a consequence of the deformation in the dispersion relation. Moreover, this correction described as a non–trivial function of the number of particles and the shape associated with the corresponding trap it could provide representative bounds for the deformation parameter ξ_1. We have proved that the deformation parameter ξ_1 can be bounded, under typical conditions, from $|\xi_1| \lesssim 10^6$ to $|\xi_1| \lesssim 10^2$, by using different classes of trapping potentials in the thermodynamic limit. In the case of a harmonic oscillator–type potential, we have obtained a bound up to $|\xi_1| \lesssim 10^4$. In references [5, 6] it was suggested the use of ultra–precise cold–atom–recoil experiments to constrain the form of the energy-momentum dispersion relation in the non–relativistic limit. There, the bound associated with ξ_1 is at least, four orders of magnitude smaller than the bound associated with a Bose–Einstein condensate trapped in a harmonic oscillator in the ideal case obtained in [15].

In a more realistic system, finite size effects and interactions among the constituents of the gas must be taken into account. To this aim, let us propose a particularly simple modified Hartree-Fock type spectrum, in the semi–classical approximation, which basically consist in the assumption that the constituents of the gas behave like non–interacting bosons moving in a self–consistent mean field, valid when the semiclassical energy spectrum ϵ_p is bigger than the associated chemical potential μ, for dilute gases [20, 21]

$$\epsilon_p = \frac{p^2}{2m} + \alpha p + U(\vec{r}) + 2U_0 n(\vec{r}), \quad (2)$$

where p is the momentum, m is the mass of the particle, and the term αp, with $\alpha = \xi_1 \frac{mc}{2M_p}$ in ordinary units, is

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the leading order modification in expression (1), with c the speed of light. The term $2U_0 n(\vec{r})$ is a mean field generated by the interactions with the other constituents of the bosonic gas, being $n(\vec{r})$ the spatial density of the cloud [20]. The coupling constant U_0 is related to the s–wave scattering length a through the following expression

$$U_0 = \frac{4\pi\hbar^2}{m}a. \qquad (3)$$

The potential term

$$U(\vec{r}) = \sum_{i=1}^{d} A_i \left|\frac{r_i}{a_i}\right|^{s_i}, \qquad (4)$$

is the generic 3–dimensional power–law potential, where A_i and a_i are energy and length scales associated with the trap [22]. Additionally, r_i are the d radial coordinates in the n_i–dimensional subspace of the 3–dimensional space. The sub–dimensions n_i satisfy the following expression in three spatial dimensions

$$\sum_{i=1}^{d} n_i = 3. \qquad (5)$$

If $d = 3$, $n_1 = n_2 = n_3 = 1$, then the potential becomes in the Cartesian trap. If $d = 2$, $n_1 = 2$ and $n_2 = 1$, then we obtain the cylindrical trap. If $d = 1$, $n_1 = 3$, then we have the spherical trap. If $s_i \to \infty$, we have a free gas in a box. The external potential included in (2) is quite general. Different combinations of these parameters give different classes of potentials, according to (4). It is noteworthy to mention that the use of these generic potentials, opens the possibility to adiabatically cool the system in a reversible way, by changing the shape of the trap [20]. The analysis of a Bose–Einstein condensate in the ideal case, weakly interacting, and with a finite number of particles, trapped in different potentials show that the main properties associated with the condensate, and in particular the condensation temperature, strongly depends on the trapping potential under consideration [22–35]. Additionally, the characteristics of the potential (in particular, the parameter that defines the shape of the potential) has a strong impact on the dependence of the condensation temperature with the number of particles (or the associated density).

The main goal of this work is to analyze the shift in the condensation temperature caused by a deformed dispersion relation in weakly interacting systems and also in systems containing a finite number of particles. We stress that these systems could be used, in principle, to obtain criteria of viability for possible signals coming from Planck scale regime, by analyzing some relevant thermodynamic variables, for instance, the number of particles, and the frequency associated with the trap, when $|\xi_1| \lesssim 1$.

II. CONDENSATION TEMPERATURE IN THE THERMODYNAMIC LIMIT; $U_0 = 0$

Due to an extensive use of some results, let us briefly summarize the results obtained in [15]. From (2), the case $U_0 = 0$ leads to

$$\epsilon_p = \frac{p^2}{2m} + \alpha p + U(\vec{r}). \qquad (6)$$

In the semiclassical approximation, the single–particle phase–space distribution may be written as [20, 21]

$$n(\vec{r}, \vec{p}) = \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1}, \qquad (7)$$

where $\beta = 1/\kappa T$, κ is the Boltzmann constant, T is the temperature, and μ is the chemical potential. The number of particles in the 3–dimensional space obeys the normalization condition [20, 21],

$$N = \frac{1}{(2\pi\hbar)^3} \int d^3\vec{r}\, d^3\vec{p}\, n(\vec{r}, \vec{p}), \qquad (8)$$

where

$$n(\vec{r}) = \int d^3\vec{p}\, n(\vec{r}, \vec{p}), \qquad (9)$$

is the spatial density. Using expression (6), and integrating expression (7) over momentum, with the help of (9), we get the spatial distribution associated with our modified semi–classical spectrum (6)

$$n(\vec{r}) = \lambda^{-3} g_{3/2}\left(e^{\beta(\mu_{eff} - U(\vec{r}))}\right) \qquad (10)$$
$$- \alpha\lambda^{-2}\left(\frac{m}{\pi\hbar}\right) g_1\left(e^{\beta(\mu_{eff} - U(\vec{r}))}\right)$$
$$+ \alpha^2\lambda^{-1}\left(\frac{m^2}{2\pi\hbar^2}\right) g_{1/2}\left(e^{\beta(\mu_{eff} - U(\vec{r}))}\right)$$

where $\lambda = \left(\frac{2\pi\hbar^2}{m\kappa T}\right)^{1/2}$, is the de Broglie thermal wavelength, $\mu_{eff} = \mu + m\alpha^2/2$ is an effective chemical potential, and $g_\nu(z)$ is the so–called Bose–Einstein function defined by [36]

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{-1}e^x - 1}. \qquad (11)$$

If we set $\alpha = 0$ in equation (10) we recover the usual result for the spatial density in the semiclassical approximation [20, 21]. By using the properties of the Bose–Einstein functions [36], assuming that $m\alpha^2/2 << \kappa T$ and integrating the normalization condition (8), we obtain an expression for the number of particles N to first order in α

$$N - N_0 = C\Pi_{l=1}^{d} A_l^{-\frac{n_l}{s_l}} a_l^{n_l} \Gamma\left(\frac{n_l}{s_l} + 1\right) \qquad (12)$$
$$\times \left[\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} g_\gamma(z)(\kappa T)^\gamma\right.$$
$$\left. - \alpha\left(\frac{m^2}{2\pi^2\hbar^3}\right) g_{\gamma-1/2}(z)(\kappa T)^{\gamma-1/2}\right],$$

where

$$\gamma = \frac{3}{2} + \sum_{l=1}^{d} \frac{n_l}{s_l}, \qquad (13)$$

is the parameter that defines the shape of the potential (4). In (12), N_0 are the particles in the ground state, $\Gamma(y)$ is the Gamma function, and C is a constant associated with the potential in question. In the case of Cartesian traps, and in consequence, for a three dimensional harmonic oscillator potential $\gamma = 3$ and $C = 8$. If we set $\alpha = 0$ in (12) then, we recover the result given in [22]. In the thermodynamic limit the conditions for condensation are given by $\mu = 0$ and $N_0 = 0$, which implies hat the Bose–Einstein functions become the corresponding Riemann Zeta functions $\zeta(x)$ [36]. Thus, expression (12) at the condensation temperature is given by

$$N = C\Pi_{l=1}^{d} A_l^{-\frac{n_l}{s_l}} a_l^{n_l} \Gamma\left(\frac{n_l}{s_l}+1\right)\left[\left(\frac{m}{2\pi\hbar^2}\right)^{3/2}\zeta(\gamma)(\kappa T_c)^{\gamma} \right. \quad (14)$$
$$\left. - \alpha\left(\frac{m^2}{2\pi^2\hbar^3}\right)\zeta(\gamma-1/2)(\kappa T_c)^{\gamma-\frac{1}{2}}\right],$$

where T_c is the condensation temperature. If we set $\alpha = 0$ in (14), we recover the usual expression for the condensation temperature T_0 for a gas trapped in a generic 3–dim power–law potential in the thermodynamic limit [22]

$$T_0 = \left[\frac{N\Pi_{l=1}^{d} A_l^{\frac{n_l}{s_l}} a_l^{-n_l}}{C\Pi_{l=1}^{d}\Gamma\left(\frac{n_l}{s_l}+1\right)}\left(\frac{2\pi\hbar^2}{m}\right)^{3/2}\right]^{1/\gamma}\frac{1}{\kappa}. \qquad (15)$$

Now, let us define

$$V_{char} = \frac{\Pi_{l=1}^{d} A_l^{\frac{n_l}{s_l}} a_l^{-n_l}}{C\Pi_{l=1}^{d}\Gamma\left(\frac{n_l}{s_l}+1\right)}, \qquad (16)$$

as the characteristic volume associated with the system. We notice that if $s_i \to \infty$ then, V_{char} becomes the volume associated with a free gas in a box (in fact the inverse of the volume with this definition). In this sense, V_{char} can be interpreted as the available volume occupied by the gas [23, 25]. At this point, it is noteworthy to mention that the most general definition of thermodynamic limit can be expressed as $N \to \infty$, $V_{char} \to 0$ keeping the product NV_{char} constant, and is valid for all power law potentials in any spatial dimensionality [25]. With the criterion given above, the condensation temperature in the thermodynamic limit is well defined. Finally, we can express the shift in the condensation temperature as a function of the number of particles N

$$\frac{T_c - T_0}{T_0} \equiv \frac{\Delta T_c}{T_0} \simeq \alpha\Omega N^{-1/2\gamma}, \qquad (17)$$

where

$$\Omega = \left(\frac{2m}{\pi}\right)^{1/2}\frac{\zeta(\gamma-1/2)}{\gamma\zeta(\gamma)}\left(\frac{V_{char}(2\hbar^2)^{3/2}}{\zeta(\gamma)}\right)^{-1/2\gamma}. \qquad (18)$$

For the sake of simplicity, let us analyze the case of spherical traps, where the corresponding potential is given by $U(r) = A_1(\frac{r}{a_1})^{s_1}$, setting $A_1 = \hbar\omega_0/2$ and $a_1 = \sqrt{\hbar/m\omega_0}$. In this case, the shift in the condensation temperature is given by

$$\frac{\Delta T_c}{T_0} \simeq \alpha\Omega_{s_1} N^{-s_1/3(s_1+2)}. \qquad (19)$$

For different values of s_1 we obtain, $\frac{\Delta T_c}{T_0} \sim \alpha N^{-1/9}$, for $s_1 = 1$, which corresponds to a linear trap. For $s_1 = 2$, $\frac{\Delta T_c}{T_0} \sim \alpha N^{-1/6}$, which is an isotropic harmonic oscillator. For $s_1 = 3$, $\frac{\Delta T_c}{T_0} \sim \alpha N^{-1/5}$. For $s_1 = 6$, $\frac{\Delta T_c}{T_0} \sim \alpha N^{-1/4}$, and so on. We notice immediately that if $s_1 \to \infty$, after some algebraic manipulation, we are able to obtain the limiting case of a bosonic gas trapped in a box

$$\frac{\Delta T_c}{T_0} \simeq \alpha\frac{2m(V\zeta(3))^{1/3}}{3\hbar}N^{-1/3}. \qquad (20)$$

Additionally, the current high precision experiments in the case of $^{39}_{19}K$, with a mass $15 \times 10^{-26} Kg$, the shift in the condensation temperature respect to the ideal result, caused by the interactions among the constituents of the gas is about 5×10^{-2} with a 1% of error [37]. These facts allows us to obtain a bound for the deformation parameter $|\xi_1| \lesssim 10^6$ for the linear trap $s_1 = 1$ (with frequencies $\omega_0/2\pi \sim 10\,Hz$ and $N \sim 10^6$), to $|\xi_1| \lesssim 10^2$ corresponding to a free gas in a box $s_1 \to \infty$, with densities about $10^{13}-10^{15}$ [20]. In fact, these bounds could be improved in a system containing massive bosons and/or lower frequencies but, where the thermodynamic limit is still valid, trapped in potentials where the parameter s_1 is sufficiently large. Here one important fact is that we are able to improve the bound associated to the deformation parameter ξ_1 by the use of different classes of potentials and it is straightforward to generalize this result to more general traps by using (17).

From (17) and (18), we notice that correction in the condensation temperature depends strongly on the functional form between the number of particles and the parameters associated to the potential in question. Finally, in the case of a harmonic oscillator potential we obtain, $\frac{\Delta T_c}{T_0} \sim \alpha N^{-1/6}$ corresponding to a shift of order $\xi_1\,10^{-6}$, which allows us to bound the parameter ξ_1 up to $|\xi_1| \lesssim 10^4$, under typical conditions.

III. WEAKLY INTERACTING MODIFIED BOSONIC GAS

Let us start with the modified Hartree–Fock spectrum (2). After some calculations, similar to the previous section, we obtain the spatial density associated to the weakly interacting case

$$n(\vec{r}) = \lambda^{-3}g_{3/2}\left(e^{\beta(\mu_{eff}-U(\vec{r})-2U_0 n(\vec{r}))}\right) \qquad (21)$$
$$- \alpha\lambda^{-2}\left(\frac{m}{\pi\hbar}\right)g_1\left(e^{\beta(\mu_{eff}-U(\vec{r})-2U_0 n(\vec{r}))}\right).$$

If we set $\alpha = 0$ in equation (21) we recover the usual result for the spatial density in the semiclassical approximation [20, 21]. By using the properties of the Bose–Einstein functions [36], we are able to expand expression (21) around $U_0 = 0$, with the result

$$n(\vec{r}) \approx n_0(\vec{r}) + U_0(2\kappa T)^{-1}\lambda^{-6}\Big[g_{3/2}(Z)g_{1/2}(Z)\Big] \quad (22)$$
$$+ \alpha U_0(2\kappa T)^{-1}\lambda^{-5}\Big(\frac{m}{\pi\hbar}\Big)\Big[g_{3/2}(Z)g_0(Z)$$
$$+ g_1(Z)g_{1/2}(Z)\Big],$$

where

$$Z = e^{\beta(\mu_{eff} - U(\vec{r}))}, \quad (23)$$

being $n_0(\vec{r})$ the space density distribution for the ideal case $U_0 = 0$,

$$n_0(\vec{r}) = \lambda^{-3}g_{3/2}(Z) - \alpha\lambda^{-2}\Big(\frac{m}{\pi\hbar}\Big)g_1(Z). \quad (24)$$

Integrating the normalization condition (8) and using expression (22) with the corresponding potential (4), allows us to obtain an expression for the number of particles as a function of the chemical potential μ, the temperature T, the coupling constant U_0, and the deformation parameter α

$$NV_{char} = \Bigg[\Big(\frac{m}{2\pi\hbar^2}\Big)^{3/2}g_\gamma(z_{eff})(\kappa T)^\gamma \quad (25)$$
$$- \alpha\Big(\frac{m^2}{2\pi^2\hbar^3}\Big)g_\gamma(z_{eff})(\kappa T)^{\gamma-1/2}$$
$$- U_0\Big(\frac{m}{2\pi\hbar^2}\Big)^3 G_{3/2,1/2,\gamma-3/2}(z_{eff})(\kappa T)^{\gamma-3/2}$$
$$+ \alpha U_0\Big(\frac{m}{2\pi\hbar^2}\Big)^{5/2}\Big(\frac{m}{\pi\hbar}\Big)(\kappa T)^\gamma$$
$$\times \Big(G_{3/2,0,\gamma-3/2}(z_{eff}) + G_{1,1/2,\gamma-3/2}(z_{eff})\Big)\Bigg],$$

where

$$G_{\eta,\sigma,\gamma-3/2}(z_{eff}) = \sum_{ij=1}^{\infty}\frac{z_{eff}^{(i+j)}}{i^\eta j^\sigma(i+j)^{\gamma-3/2}}, \quad (26)$$

and we have defined an effective fugacity

$$z_{eff} = e^{\beta(\mu+m\alpha^2/2)}. \quad (27)$$

In order to obtain the leading correction on the condensation temperature caused by the interactions in our deformed bosonic gas, let us expand (25) to first order in $T = T_0$, $\mu = 0$, $U_0 = 0$, and $\alpha = 0$. Recalling that T_0 is the condensation temperature in the thermodynamic

limit given by expression (15), with the result

$$NV_{char} = \Bigg[\Big(\frac{m}{2\pi\hbar^2}\Big)^{3/2}\zeta(\gamma)(\kappa T_0)^\gamma \quad (28)$$
$$+ [T - T_0]\Big(\frac{m}{2\pi\hbar^2}\Big)^{3/2}\gamma\zeta(\gamma)\kappa(\kappa T_0)^{\gamma-1}$$
$$- U_0\Big(\frac{m}{2\pi\hbar^2}\Big)^3 G_{3/2,1/2,\gamma-3/2}(1)(\kappa T_0)^{\gamma+1/2}$$
$$+ \mu\Big(\frac{m}{2\pi\hbar^2}\Big)^{3/2}\zeta(\gamma-1)(\kappa T_0)^{\gamma-1}$$
$$- \alpha\frac{m^2}{\pi^2\hbar^3}\zeta(\gamma-1/2)(\kappa T_0)^{\gamma-1/2}\Bigg].$$

At the condensation temperature T_c for large N, in the mean field approach the chemical potential takes the value $\mu_c = 2\,U_0 n_0(\vec{r} = \vec{0})$ [20, 21]. In the usual case $\alpha = 0$, $n_0(\vec{r} = \vec{0}) = \lambda_c^{-3}\zeta(3/2)$ in the large N limit, which means that the critical density at the center of the trap is the same as that of the uniform model [20]. However, in our case, we have to modified the value of μ_c at the condensation temperature according to expression (24), due to the divergent behavior of the Bose-Einstein functions related to $n_0(\vec{r} = \vec{0})$. When the integrals associated with the Bose–Einstein functions converges, the value $m\alpha^2/2$ is negligible and can be replaced by zero. Nevertheless, when the integral associated to the Bose–Einstein functions can diverge at $Z \to 1$ the minimum of the energy associated with the system must be taken into account [25]. In this section we are interested in the corrections due to α in the large N limit, so we will take as the minimum of energy in the system $m\alpha^2/2$. Let us define $n_0(\vec{r} = \vec{0})$ at the condensation temperature using expression (24) as follows

$$n_0(\vec{r} = \vec{0}) = \lambda_c^{-3}g_{3/2}(e^{\beta_c m\alpha^2/2}) \quad (29)$$
$$- 2\alpha U_0\lambda_c^{-2}\Big(\frac{m}{\pi\hbar}\Big)g_1(e^{\beta_c m\alpha^2/2}).$$

We can define the Bose–Einstein functions $g_{3/2}(e^{\beta_c m\alpha^2/2})$ and $g_1(e^{\beta_c m\alpha^2/2})$, when $(\beta_c m\alpha^2/2) \to 0$ as [36]

$$g_{3/2}(e^{\beta_c m\alpha^2/2}) \simeq \zeta(3/2) + \Gamma(-1/2)\Big(\frac{m\alpha^2}{2\kappa T_c}\Big)^{1/2} \quad (30)$$

$$g_1(e^{\beta_c m\alpha^2/2}) \simeq \ln\Big(\frac{2\kappa T_c}{m\alpha^2}\Big). \quad (31)$$

Neglecting second order terms in U_0 and α, this allows us to write μ_c using expression (29) as follows

$$\mu_c \simeq 2U_0\lambda_c^{-3}\zeta(3/2) - 2\alpha U_0\lambda_c^{-2}\Big(\frac{m}{\pi\hbar}\Big)\ln\Big(\frac{2\kappa T_c}{m\alpha^2}\Big). \quad (32)$$

If we take the limit $\alpha \to 0$, then we recover the usual value for μ_c at the condensation temperature [20, 21]. Inserting (32) in (28), we finally obtain the shift in the

condensation temperature in function of the number of particles

$$\frac{\Delta T_c}{T_0} \simeq -(aR_0)^{\frac{1}{2\gamma}}\left(\frac{m\Lambda^2}{2\pi\hbar^2}\right)^{\frac{1}{2}} N^{\frac{1}{2\gamma}} \tag{33}$$
$$+ \alpha\frac{(8\pi m)^{1/2}\zeta(\gamma - 1/2)}{\zeta(\gamma)\gamma}(R_0 N)^{-\frac{1}{2\gamma}}$$
$$+ \alpha a\frac{4m\zeta(\gamma - 1)}{\pi\hbar\zeta(\gamma)\gamma}\ln\left(\frac{(R_0 N)^{\frac{1}{\gamma}}}{m\alpha^2}\right),$$

where

$$\Lambda = \frac{2\zeta(3/2)\zeta(\gamma - 1) - G_{3/2,1/2,\gamma-3/2}(1)}{\zeta(\gamma)\gamma}, \tag{34}$$

$$R_0 = \left(\frac{2\pi\hbar^2}{m}\right)^{3/2}\left[\frac{V_{char}}{\zeta(\gamma)}\right]. \tag{35}$$

Setting $\alpha = 0$ in equation (33) we recover the usual shift on the condensation temperature caused by weakly interactions. Let us analyze the case of spherical traps $U(r) = A_1(\frac{r}{a_1})^{s_1}$, together with $A_1 = \hbar\omega_0/2$ and $a_1 = \sqrt{\hbar/m\omega_0}$. Notice that the possibility of detecting the term depending upon the deformation parameter effect (δT_c^α) requires that, if $\delta T_c^{(0)}$ is the experimental error related to the measurement of the condensation temperature when $\alpha = 0$, then $\delta T_c^{(0)} < |\delta T_c^\alpha|$. In our case this entails

$$\delta T_c^{(0)} < \left|\alpha\frac{(8\pi m)^{1/2}\zeta(\gamma - 1/2)}{\zeta(\gamma)\gamma}(R_0 N)^{-\frac{1}{2\gamma}}\right. \tag{36}$$
$$\left.+ \alpha a\frac{4m\zeta(\gamma - 1)}{\pi\hbar\zeta(\gamma)\gamma}\ln\left(\frac{(R_0 N)^{\frac{1}{\gamma}}}{m\alpha^2}\right)\right|.$$

For spherical traps $\gamma = 3(s_1 + 2)/2s_1$. The shift in the condensation temperature caused by interactions is typically 5×10^{-2}, with a 1% of error [37, 38], then from expression (36) and the results given above, in the case of $^{39}_{19}K$, with a mass 15×10^{-26} kg, $a \sim 10^{-9}$m, and $\omega_0 \sim 10$Hz, allows us to obtain a criterion on $|\delta T_c^\alpha|$ as a function of the number of particles when $|\xi_1| \lesssim 1$. For different values of the shape parameter γ we obtain, $N > 10^{33}$ for $s_1 = 1$, $N > 10^{22}$ for $s_1 = 2$, $N > 10^{17}$ for $s_1 = 4$, $N > 10^{13}$ for $s_1 = 9$, $N > 10^{11}$ for $s_1 = 18$ and so on. Notice that expression (36) is not valid for the case $s_1 = 6$, (which implies $\gamma = 2$) due to the divergent behavior of $\zeta(1)$. This special case, $s_1 = 6$, defines a limit between a positive shift and a negative one, caused by the deformation parameter α. In other words, the shift caused by the deformation parameter is positive when $s_1 < 6$, for a positive α. Conversely, with $s_1 > 6$ the corresponding shift is negative. Notice that, if the parameter s_1 is sufficiently large then, the number of particles decreases, but the shift on the condensation temperature caused by the deformation parameter becomes negative.

On the other hand, keeping the number of particles fixed, with say $N \sim 10^5 - 10^6$, we obtain frequencies of order $\omega_0 \sim 10^{-9}$Hz for $s_1 = 1$, $\omega_0 \sim 10^{-10}$Hz for $s_1 = 2$, $\omega_0 \sim 10^{-15}$Hz for $s_1 = 4$ and so on. In other words, for a fixed number of particles, $\omega_0 \to 0$ if the parameter s_1 grows.

In the case of an anisotropic three-dimensional harmonic oscillator potential $\gamma = 3$, with $A_l = \hbar\omega_l/2$, $a_l = \sqrt{\hbar/m\omega_l}$, then from expression (33) we obtain

$$\frac{\Delta T_c}{T_0} \simeq -\left(\frac{a}{a_{ho}}\right)\left[\frac{2\zeta(3/2)\zeta(2) - G_{3/2}(1)}{(2\pi)^{1/2}3\zeta(3)^{5/6}}\right]N^{1/6} \tag{37}$$
$$+ \alpha\frac{2^{3/2}\pi^{-1/2}\zeta(5/2)}{3\zeta(3)^{5/6}}\left(\frac{m}{\hbar\bar\omega}\right)^{1/2}N^{-1/6}$$
$$+ \alpha a\left(\frac{4m\zeta(2)}{3\pi\hbar\zeta(3)}\right)\ln\left(\frac{2(\hbar\bar\omega)N^{1/3}}{\zeta(3)m\alpha^2}\right),$$

where we have used the usual definitions $\bar\omega = (\omega_1\omega_2\omega_3)^{1/3}$ and $a_{ho} = \left(\frac{\hbar}{m\bar\omega}\right)^{1/2}$. In this case, $N > 10^{22}$ which corresponds to a shift on the condensation temperature of order $\sim 10^{-5}$, positive. Conversely, $\bar\omega \sim 10^{-10}$Hz, for a fixed $N \sim 10^5 - 10^6$ corresponding to a shift on the condensation temperature of order 10^{-3}.

It is noteworthy to mention that ξ_1 could be bounded up to $|\xi_1| \lesssim 10^3$ by using (37) for $N \sim 10^5 - 10^6$, and $\bar\omega \sim 10$Hz, that is, one order of magnitude less than the bound obtained in [15], which is notable. If we set $\alpha = 0$ in expression (37) we recover the result given in [29].

IV. DEFORMED BOSONIC GAS AND FINITE SIZE CORRECTIONS

In this section let us calculate the leading correction on the condensation temperature caused by finite size effects in our modified bosonic gas. The correction to the condensation temperature originates in the zero–point motion, or equivalently, at the associated ground–state energy ϵ_0 [21, 22]. Thus, at the condensation temperature the chemical potential is given by $\mu = \epsilon_0$. Setting $\mu = \epsilon_0$ and $a = 0$ in (28), we obtain that the relative shift in the condensation temperature is given by

$$\frac{\Delta T_c}{T_0} = -\epsilon_0\frac{\zeta(\gamma - 1)}{\zeta(\gamma)}\left(NV_{char}\left(\frac{2\pi\hbar^2}{m}\right)^{3/2}\right)^{-1/\gamma} \tag{38}$$
$$+ \alpha\left(\frac{2m}{\pi}\right)^{1/2}\frac{\zeta(\gamma - 1/2)}{\zeta(\gamma)}\left(NV_{char}\left(\frac{2\pi\hbar^2}{m}\right)^{3/2}\right)^{-1/2\gamma}$$
$$+ O(\epsilon_0^2, \alpha^2).$$

For spherical traps, setting $\epsilon_0 = c_1\hbar\omega_0$ [32], the shift on the condensation temperature can be expressed as follows

$$\frac{\Delta T_c}{T_0} \simeq -c_1\hbar\omega_0\frac{\zeta(\gamma - 1)}{\zeta(\gamma)}(\Omega_{s_1}N)^{-2s_1/3(s_1+2)} \tag{39}$$
$$+ \alpha\left(\frac{2m}{\pi}\right)^{1/2}\frac{\zeta(\gamma - 1/2)}{\zeta(\gamma)}(\Omega_{s_1}N)^{-s_1/3(s_1+2)}.$$

where

$$\Omega_{s_1} = V_{char_{s_1}} \left(\frac{2\pi\hbar^2}{m}\right)^{3/2}. \tag{40}$$

For different values of the shape parameter γ, we obtain from expression (39), for instance, in the case of linear traps $s_1 = 1$, $\frac{\Delta T_c}{T_0} \sim \epsilon_{0_{s_1=1}} N^{-2/9} + \alpha N^{-1/9}$. For $s_1 = 2$, which corresponds to an isotropic harmonic oscillator, $\frac{\Delta T_c}{T_0} \sim \epsilon_{0_{s_1=2}} N^{-1/3} + \alpha N^{-1/6}$. For $s_1 = 3$, $\frac{\Delta T_c}{T_0} \sim \epsilon_{0_{s_1=3}} N^{-2/5} + \alpha N^{-1/5}$. For $s_1 = 4$, $\frac{\Delta T_c}{T_0} \sim \epsilon_{0_{s_1=4}} N^{-4/9} + \alpha N^{-2/9}$, and so on. The possibility of detecting the term depending upon the deformation parameter effect entails in this case

$$\delta T_c^{(0)} < \left| \alpha \left(\frac{2m}{\pi}\right)^{1/2} \frac{\zeta(\gamma - 1/2)}{\zeta(\gamma)} (\Omega_{s_1} N)^{-s_1/3(s_1+2)} \right|. \tag{41}$$

The shift in the condensation temperature caused by finite size effects is typically of order 10^{-2} [21], then from expression (41) and the results given above, in the case of $^{39}_{19}K$ and $\omega_0 \approx 10$Hz, leads to $N > 10^{33}$ for $s_1 = 1$, $N > 10^{17}$ for $s_1 = 4$, $N > 10^{14}$ for $s_1 = 6$, $N > 10^{13}$ for $s_1 = 9$, and so on. Conversely, keeping the number of particles fixed with, let say $N \sim 10^3 - 10^6$, we obtain, $\omega_0 \sim 1.62 \times 10^{-9}$Hz for $s_1 = 1$, $\omega_0 \sim 8.70 \times 10^{-10}$Hz for $s_1 = 2$, $\omega_0 \sim 5.70 \times 10^{-10}$Hz for $s_1 = 4$ and so on. Here the parameter s_1 has the same behavior as in the interacting case, that is, $\omega_0 \to 0$ implies large values for s_1, when the number of particles is fixed.

For an anisotropic three-dimensional harmonic oscillator potential ($\gamma = 3$), we obtain that the relative correction in the condensation temperature is given by

$$\frac{\Delta T_c}{T_0} = -\frac{\zeta(2)}{3\zeta(3)^{2/3}} \frac{\epsilon_0}{\hbar\bar{\omega}} N^{-1/3} \tag{42}$$
$$+ \alpha \frac{\zeta(5/2)}{3\zeta(3)^{5/6}} \left(\frac{8m}{\pi\hbar\bar{\omega}}\right)^{1/2} N^{-1/6} + O(\epsilon_0^2, \alpha^2).$$

In this case, $N > 10^{22}$ which corresponds to a shift on the condensation temperature of order $\sim 10^{-10}$. Additionally, keeping $N \sim 10^3 - 10^6$ implies $\bar{\omega} \sim 8.70 \times 10^{-10}$Hz, corresponding to a shift of order 10^{-8}. If we set $\alpha = 0$ then, we recover the usual result [21, 28, 30, 34].

V. CONCLUSIONS

Using the formalism of the semiclassical approximation, we have analyzed the Bose–Einstein condensation for a modified bosonic gas trapped in a 3–D power law potential in three regimes, namely, the thermodynamic limit, finite size systems, and weakly interacting systems. We have deduced the shift on the condensation temperature in the thermodynamic limit, in a weakly interacting systems, and finite size systems as well, in function of the number of particles and the trap parameters, which are valid for any potential defined by the generic

3–dimensional power–law potential (4) within the semiclassical approximation. We have obtained a bound up to $|\xi_1| \lesssim 10^6$ for linear traps to $|\xi_1| \lesssim 10^2$ corresponding to a free gas in a box, and $|\xi_1| \lesssim 10^4$ for harmonic oscillator type potential in the ideal case, under typical conditions. We stress here that an improvement of the precision in the condensation temperature measurement would also allow to improve the bounds on ξ_1.

For weakly interacting systems, we have obtained for the case $|\xi_1| \lesssim 1$, that if the trap parameter s_1 is sufficiently large then, this decreases the number of particles, but lead to corrections on the condensation temperature of order 10^{-5} for any trap parameter s_1, which is approximately 3 orders of magnitude smaller than the typical correction 10^{-2}. Conversely, keeping the number of particles $N \sim 10^5 - 10^6$ fixed, leading to frequencies of order $10^{-15} - 10^{-9}$ corresponding to shifts on the condensation temperature up to $10^{-5} - 10^{-3}$, for different values of the shape parameter s_1. These problems could be solved, in principle, just tuning the interaction coupling by Feshbach resonances to very small values of the scattering length a, that is, almost to the ideal case, and then, reducing the contribution of interactions on the condensation temperature below the Planck-scale induced shift for a sufficient large parameter s_1. Nevertheless, these facts could affect the thermodynamical equilibrium of the system, involving some technical difficulties.

On the other hand, finite size effects for sufficient large s_1 leads to a very small correction on the condensation temperature of order 10^{-10}, for any trap parameter s_1 and fixed frequency ω_0, which is 8 orders of magnitude smaller than the typical correction 10^{-2}. It is a condition impossible to fulfill. Conversely, keeping the number of particles $N \sim 10^5 - 10^6$ fixed, lead to frequencies up to $10^9 - 10^{-15}$ corresponding to shifts on the condensation temperature of order $10^{-8} - 10^{-5}$, for different values of the shape parameter s_1. In other words, these facts suggest that finite size effects are technologically impossible to be tuned below Planck-scale induced effects, at least for current experiments.

For fixed frequencies of order 10Hz, in the case of a harmonic oscillator potential, we obtain $N > 10^{22}$, which implies a shift on the condensation temperature of order 10^{-5}, in a weakly interacting system, and a shift on the condensation temperature of order 10^{-10}, in finite size systems. Conversely, for a fixed number of particles $N \sim 10^5 - 10^6$, this leads to a shift of order 10^{-3} for weakly interacting systems, with $\bar{\omega} \sim 10^{-10}$. For finite size systems, $\omega \sim 10^{-8}$ corresponding to shifts of order 10^{-8}. Notice that the relevant contributions coming from the product $(N\omega^n)^m$, where m and n, depend on the properties of the trap in question. These facts suggest that many–body contributions on the relevant thermodynamic functions associated with the condensate could be used, in principle, to constrain significantly the parameter ξ_1, which in our case, for instance, could be bounded

up to $|\xi_1| \lesssim 10^3$ by using (37), under typical conditions.

Here is important to emphasize that the possibility of a systematic error in the measurements due to the variation in the corresponding trap frequency, could affect the usual predictions on the corresponding shift in the condensation temperature. However, these systematic errors can be estimated to be less than 1% [37, 38], or even less than 0.5%, as it was reported in [39]. In fact, in reference [37] each measurement at a given s–wave scattering, is compared with a reference measurement for small values of this parameter of order ~ 0.005, with the same frequency ω and an approximately equal number of particles. Thus, under these circumstances, the relative shift $\Delta T_c/T_0$ depends only on the s–wave scattering length effects, that is, is assumed to be unaffected from all independent effects, including systematic errors in the absolute calibration of N and finite-size effects. In addition, systematic errors are often more easily controlled at lower temperatures.

On the other hand, the relative shift $\Delta T_c/T_0$ caused by interactions, is highly trap–dependent, as can be seen from expression (33). For instance, in the case of harmonic traps, long–range fluctuations are suppressed [24] and the leading term in the relative shift $\Delta T_c/T_0$ can be calculated with perturbative methods [40] like in the present report. However, higher orders in the relative shift $\Delta T_c/T_0$, calculated by using a non–perturbative approach behave as $b_1\delta' + (b_2' \ln \delta' + b_2'')\delta'^2$, in the case of a harmonic oscillator potential, where $\delta' \equiv a/\lambda$ with λ the thermal de Broglie wavelength [41]. Here a good fit [24] yields $b_1 \simeq -3.426$, $b_2' \simeq -45.86$ and $b_2'' \simeq -155.0$. Thus, it could be interesting to look at the corrections in the relative shift on the condensation temperature caused by the deformation parameters by using a non–perturbative approach.

Finally, we must add that the possible detection of these corrections, could be out of the current technology. Nevertheless, it is remarkable that an adequate choice of the shape associated with the potential under consideration, together with the many–body contributions, open the possibility of planning specific scenarios that could be used, in principle, to obtain a possible measure of the effects caused by the quantum structure of space–time.

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