

Causal Cones, Cone Preserving Transformations and Causal Structure in Special and General Theory of Relativity*

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Abstract We present a short review of geometric and algebraic approach to causal cones and describe cone preserving transformations and their relationship with causal structure related to special and general theory of relativity. We describe Lie groups, especially matrix Lie groups, homogeneous and symmetric spaces and causal cones and certain implications of these concepts in special and general theory of relativity related to causal structure and topology of space-time. We compare and contrast the results on causal relations with those in the literature for general space-times and compare these relations with K-causal maps. We also describe causal orientations and their implications for space-time topology and discuss some more topologies on space-time which arise as an application of domain theory.

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Key Words: Space - time manifolds; causal cones; causal structure; domain theory.

1 Introduction

The notion of causal order is a basic concept in physics and in the theory of relativity in particular. A space time metric determines causal order and causal cone structure. Alexandrov [1, 2] proved that a causal order can determine a topology of space-time called Alexandrov topology which, as is now well known, coincides with manifold topology if the space time is strongly causal. The books by Hawking-Ellis, Wald and Joshi [3, 4, 5] give a detailed treatment of causal structure of space-time. However, while general relativity employs a Lorentzian metric, all genuine approaches to quantum gravity are free of space-time metric. Hence the question arises whether there exists a structure which gets some features of causal cones (light cones) in a purely topological or order-theoretic manner. Motivated by the requirement on suitable structures for a theory of quantum gravity, new notions of causal structures and cone structures were deployed on a space-time.

The order theoretic structures, namely causal sets have been extensively used by Sorkin and his co-workers in developing a new approach to quantum gravity [6]. As a part of this program, Sorkin and Woolgar [7] introduced a relation called K - causality and proved interesting results by making use of Vietoris topology. Based on this work and other recent work, S. Janardhan and R.V.Saraykar [8, 9] and E.Minguzzi [10, 11] proved many interesting results. Especially after good deal of effort, Minguzzi [10] proved that K - causality condition is equivalent to *stably causal* condition.

More recently, K.Martin and Panangaden [12] making use of domain theory, a branch of theoretical computer science, proved fascinating results in the causal structure theory of space-time. The

remarkable fact about their work is that only *order* is needed to develop the theory and topology is an outcome of the *order*. In addition to this consequence, there are abstract approaches, algebraic as well as geometric to the theory of cones and cone preserving mappings. Use of quasi-order (a relation which is reflexive and transitive) and partial order is made in defining the cone structure. Such structures and partial orderings are used in the optimization problems [13], game theory and decision making etc [14]. The interplay between ideas from theoretical computer science and causal structure of space-time is becoming more evident in the recent works [15, 16].

Keeping in view these developments, in this paper, we present a short review of geometric and algebraic approach to causal cones and describe cone preserving transformations and their relationship with causal structure. We also describe certain implications of these concepts in special and general theory of relativity related to causal structure and topology of space-time.

Thus in section 2, we begin with describing Lie groups, especially matrix Lie groups, homogeneous spaces and then causal cones. We give an algebraic description of cones by using quasi-order. Furthermore, we describe cone preserving transformations. These maps are generalizations of causal maps related to causal structure of space-time which we shall describe in section 3. We then describe explicitly Minkowski space as an illustration of these concepts and note that some of the space-time models in general theory of relativity can be described as homogeneous spaces.

In section 3, we describe causal structure of space time, causality conditions, K-causality and hierarchy among these conditions in the light of recent work of Minguzzi and Sanchez [17]. We also describe geometric structure of causal group, a group of transformations preserving causal structures or a group of causal maps on a space-time.

In section 4, we describe causal orientations and their implications for space-time topology. We find a parallel between these concepts and concepts developed by Martin and Panangaden [12] to describe topology of space time, especially a globally hyperbolic one. Finally we discuss some more topologies on space-time which arise as an application of domain theory.

We end the paper with concluding remarks.

2 Causal Cones and cone preserving transformations

To begin with, we describe Lie groups, matrix Lie groups, homogeneous and symmetric spaces and state some results about them. These will be used in the discussion on causal cones. We refer to the books [18, 19, 20] for more details.

Definition : Lie groups and matrix Lie groups:

Lie group: A finite dimensional manifold G is called a Lie group if G is a group such that the group operations, composition and inverse are compatible with the differential structure on G .

This means that the mappings

$$G \times G \rightarrow G : (x, y) \mapsto x.y \text{ and}$$

$$G \rightarrow G : x \mapsto x^{-1}$$

are C^∞ as mappings from one manifold to other.

The n -dimensional real Euclidean space R^n , n -dimensional complex Euclidean space C^n , unit sphere S^1 in R^2 , the set of all $n \times n$ real matrices $M(n, R)$ and the set of all $n \times n$ complex matrices $M(n, C)$ are the simplest examples of Lie groups. $M(n, R)$ (and $M(n, C)$) have subsets which are Lie groups in their own right. These Lie groups are called *matrix Lie groups*. They are important because most of the Lie groups appearing in physical sciences such as classical and quantum mechanics, theory of relativity - special and general, particle physics etc are matrix Lie groups. We describe some of them here, which will be used later in this article.

Gl(n,R) : General linear group of $n \times n$ real invertible matrices. It is a Lie group and topologically an open subset of $M(n, R)$. Its dimension is n^2 .

Sl(n,R) : Special linear group of $n \times n$ real invertible matrices with determinant +1. It is a closed subgroup of $Gl(n, R)$ and a Lie group in its own right, with dimension $n^2 - 1$.

O(n) : Group of all $n \times n$ real orthogonal matrices. It is called an orthogonal group. It is a Lie

group of dimension $\frac{n(n-1)}{2}$.

SO(n): Special orthogonal group- It is a connected component of $O(n)$ containing the identity I and also a closed (compact) subgroup of $O(n)$ consisting of real orthogonal matrices with determinant +1. In particular $SO(2)$ is isomorphic to S^1 .

The corresponding Lie groups which are subsets of $M(n, C)$ are $GL(n, C)$, $SL(n, C)$, $U(n)$ and $SU(n)$ respectively, where *orthogonal* is replaced by *unitary*. $SU(n)$ is a compact subgroup of $GL(n, C)$. For $n=2$, it can be proved that $SU(2)$ is isomorphic to S^3 , the unit sphere in R^4 . Thus S^3 is a Lie group.[However for topological reasons, S^2 is not a Lie group, though it is C^∞ -differentiable manifold]

O(p,q) and SO(p,q) : Let p and q be positive integers such that $p + q = n$. Consider the quadratic form $Q(x_1, x_2, \dots, x_n)$ given by $Q = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 \dots - x_n^2$.

The set of all $n \times n$ real matrices which preserve this quadratic form Q is denoted by $O(p, q)$ and a subset of $O(p, q)$ consisting of those matrices of $O(p, q)$ whose determinant is +1, is denoted by $SO(p, q)$. Both $O(p, q)$ and $SO(p, q)$ are Lie groups. Here preserving quadratic form Q means the following:

Consider standard inner product η on $R^{p+q} = R^n$ given by the diagonal matrix:

$\eta = \text{diag}(1, 1 \dots 1, -1, -1 \dots -1)$, (1 appearing p times).

Then η gives the above quadratic form $Q(x_1, x_2, \dots, x_n)$,

i.e. $X\eta X^T = Q(x_1, x_2, \dots, x_n)$ where $X = [x_1, x_2, \dots, x_n]$. $n \times n$ matrix A is said to preserve the quadratic form Q if $A^T \eta A = \eta$.

$O(p, q)$ is called *indefinite orthogonal group* and $SO(p, q)$ is called *indefinite special orthogonal group*. Dimension of $O(p, q)$ is $\frac{n(n-1)}{2}$.

Assuming both p and q are nonzero, neither of the groups $O(p, q)$ or $SO(p, q)$ are connected. They have respectively four and two connected components. The identity component of $O(p, q)$ is denoted by $SO_o(p, q)$ and can be identified with the set of elements in $SO(p, q)$ which preserves both orientations.

In particular $O(1, 3)$ is the Lorentz group, the group of all Lorentz transformations, which is of central importance for electromagnetism and special theory of relativity. $U(p, q)$ and $SU(p, q)$ are defined similarly. For more details, we refer the reader to [18, 21]

We now define Homogeneous spaces and discuss some of their properties:

Definition : We say that a Lie group G is *represented as a Lie group of transformations of a C^∞ manifold M* (or has a left (Lie)- action on M) if to each $g \in G$, there is associated a diffeomorphism from M to itself: $x \mapsto \psi_g(x), x \in M$ such that $\psi_{gh} = \psi_g \psi_h$ for all $g, h \in G$ and $\psi_e = Id.$, Identity map of M , and if further-more $\psi_g(x)$ depends smoothly on the arguments g, x . i.e. the map $(g, x) \mapsto \psi_g(x)$ is a smooth map from $G \times M \rightarrow M$.

The Lie group G is said to have a *right action* on M if the above definition is valid with the property $\psi_g \psi_h = \psi_{gh}$ replaced by $\psi_g \psi_h = \psi_{hg}$.

If G is any of the matrix Lie groups

$GL(n, R), O(n, R), O(p, q)$ or

$GL(n, C), U(n), U(p, q)$ (where $p + q = n$), then G acts in the obvious way on the manifold R^n or $R^{2n} = C^n$. In these cases, the elements of G act as linear transformations.

The action of a group G is said to be *transitive* if for every two points x, y of M , there exists an element $g \in G$ such that $\psi_g(x) = y$.

Definition : A manifold on which a Lie group acts transitively is called a *homogeneous space* of the Lie group.

In particular, any Lie group G is a homogeneous space for itself under the action of left multiplication. Here G is called the *Principal left homogenous space* (of itself). Similarly the action $\psi_g(h) = hg^{-1}$ makes G into its own *Principal right homogeneous space*.

Let x be any point of a homogeneous space of a Lie group G . The *isotropy group* (or stationary group) H_x of the point x is the stabilizer of x under the action of G : $H_x = \{g \in G / \psi_g(x) = x\}$.

We have the following lemma.

Lemma : All isotropy groups H_x of points x of a homogeneous space are isomorphic.

Proof : Let x, y be any two points of the homogeneous space. Let $g \in G$ be such that $\psi_g(x) = y$. Then the map $H_x \rightarrow H_y$ defined by $h \mapsto ghg^{-1}$ is an isomorphism. (Here we have assumed the left action).

We thus denote simply by H , the isotropy group of some (and hence of every element modulo isomorphism) element of M on which G acts on the left.

We now have the following theorem.

Theorem : There is a one- one correspondence between the points of a homogeneous space M of the Lie group G , and the left cosets gH of H in G , where H is the isotropy group and G is assumed to act on the left.

Proof: Let x_0 be any point of the manifold M . Then with each left coset gH_{x_0} we associate the point $\psi_g(x_0)$ of M . Then this correspondence is well- defined, i.e. independent of the choice of representative of the coset, one - one and onto.

It can be shown under certain general conditions that the isotropy group H is a closed sub group of G , and the set G/H with the natural quotient topology can be given a unique (real) analytic manifold structure such that G is a Lie transformation group of G/H . Thus $M \approx G/H$.

Examples of homogeneous spaces are:

1. Stiefel manifolds : For each $n, k(k \leq n)$, the Stiefel manifold $V_{n,k}$ has as its points all orthonormal frames $x = (e_1, e_2, \dots, e_k)$ of k vectors in Euclidean n -space i.e. ordered sequences of k orthonormal vectors in R^n . Then $V_{n,k}$ is embeddable as a non- singular surface of dimension $nk - k(k + 1)/2$ in R^{nk} and can be visualized as $SO(n)/SO(n - k)$. In particular we have $V_{n,n} \cong O(n), V_{n,n-1} \cong SO(n), V_{n,1} \cong S^{n-1}$.

2. Grassmannian manifolds : The points of the Grassmannian manifold $G_{n,k}$, are by definition, the k - dimensional planes passing through the origin of n -dimensional Euclidean space. This is a smooth manifold and it is given by

$$G_{n,k} \cong O(n)/O(k) \times O(n - k).$$

We now define symmetric spaces.

Definition : A simply connected manifold M with a metric g_{ab} defined on it, is called a *symmetric space* (*symmetric manifold*) if for every point x of M , there exists an isometry (motion) $s_x : M \rightarrow M$ with the properties that x is an isolated fixed point of it, and that the induced map on the tangent space at x reflects (reverses) every tangent vector at x i.e. $\xi \mapsto -\xi$. Such an isometry is called a *symmetry of M of the point x* .

For every symmetric space, covariant derivative of Riemann curvature tensor vanishes.

For a homogeneous symmetric manifold M , let G be the Lie group of all isometries of M and let H be the isotropy group of M with respect to left action of G on M . Then , as we have seen above, M can be identified with G/H , the set of left cosets of H in G . As examples of such spaces in general relatively, we have the following space-times:

Space of constant curvature with isotropy group $H = SO(1, 3)$:

1. Minkowski space R^4 .

2. The de Sitter space

$S_+ = SO(1, 4)/SO(1, 3)$. Here S_+ is homeomorphic to $R \times S^3$ and the curvature tensor R is the identity operator on the space of bivectors $\Lambda^2(R^4), R = Id$.

3. The anti- de Sitter space

$S = SO(2, 3)/SO(1, 3)$. This space is homeomorphic to $S^1 \times R^3$ and its universal covering space is homeomorphic to R^4 . Here curvature tensor $R = -Id$.

Another example of symmetric space-time is the symmetric space M_t of plane waves. For these spaces the isotropy group is abelian, and the isometry group is soluble (*solvable*). (A group G is called *solvable* if it has a finite chain of normal subgroups $\{e\} < G_1 < \dots < G_r = G$, beginning with the identity subgroup and ending with G , all of whose factors G_{i+1}/G_i are abelian). In terms of suitable coordinates, the metric has the form

$ds^2 = 2dx_1 dx_4 + [(cos t)x_2^2 + (sin t)x_3^2] dx_4^2 + dx_2^2 + dx_3^2$, $cos t \geq sin t$. The curvature tensor is constant (refer [20]).

Gödel universe [3] is also an example of a homogeneous space but it is not a physically reasonable model since it contains closed time like curve through every point. We now turn our attention to Causal cones and cone preserving transformations.

We note that all genuine approaches to quantum gravity are free of space-time metric while general relativity employs a Lorentzian space-time metric. Hence, the question arises whether there exists a structure which gets some features of light cones in a purely topological manner. Motivated by the requirements on suitable structures for a theory of quantum gravity, new notions of causal structure and cone structures were developed on a space-time M . Here we describe these notions.

The definition of *causal cone* is given as follows:

Let M be a finite dimensional real Euclidean vector (linear) space with inner product \langle, \rangle . Let R^+ be the set of positive real numbers and $R_0^+ = R^+ \cup \{0\}$. A subset C of M is a *cone* if $R^+C \subset C$ and is a *convex cone* if C , in addition, is a convex subset of M . This means, if $x, y \in C$ and $\lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y \in C$. In other words, C is a convex cone if and only if for all $x, y \in C$ and $\lambda, \mu \in R^+$, $\lambda x + \mu y \in C$. We call cone C as *non-trivial* if $C \neq -C$. If C is non-trivial, then $C \neq \{0\}$ and $C \neq M$.

We use the following notations:

- i. $M^c = C \cap -C$
- ii. $\langle C \rangle = C - C = \{x - y / x, y \in C\}$
- iii. $C^* = \{x \in M / \forall y \in C, (x, y) \geq 0\}$

Then M^c and $\langle C \rangle$ are vector spaces. They are called the *edge* and the *span* of C . The set C^* is a closed convex cone called the *dual cone* of C . This definition coincides with the usual definition of the dual space M^* of M by using inner product $(,)$. If C is a closed convex cone, we have $C^{**} = C$, and $(C^* \cap -C^*) = \langle C \rangle^\perp$, where for $U \subset M$, $U^\perp = \{y \in M / \forall u \in U, (u, y) = 0\}$.

Definition : Let C be a convex cone in M . Then C is called *generating* if $\langle C \rangle = M$. C is called *pointed* if there exists a $y \in M$ such that for all $x \in C - \{0\}$, we have $(x, y) > 0$. If C is closed, it is called *proper* if $M^c = \{0\}$. C is called *regular* if it is generating and proper. Finally, C is called *self-dual*, if $C^* = C$.

If M is an ordered linear space, the Clifford's theorem [22] states that M is directed if and only if C is generating.

The set of interior points of C is denoted by C° or $int(C)$. The interior of C in its linear span $\langle C \rangle$ is called the *algebraic interior* of C and is denoted by $alg\ int(C)$.

Let $S \subset M$. Then the closed convex cone generated by S is denoted by $Cone(S)$:

$Cone(S) =$

closure of $\left\{ \sum_{finite} r_s s / s \in S, r_s \geq 0 \right\}$.

If C is a closed convex cone, then its interior C° is an open convex cone. If Ω is an open convex cone, then its closure $\bar{\Omega} = cl(\Omega)$ is a closed convex cone. For an open convex cone, we define the dual cone by

$$\Omega^* = \{x \in v / \forall y \in \bar{\Omega} - \{0\} (x, y) > 0\} = int(\bar{\Omega}^*) .$$

If $\bar{\Omega}$ is proper, we have $\Omega^{**} = \Omega$

We now have the following results: (cf [23])

Proposition : Let C be a closed convex cone in M . Then the following statements are equivalent:

- i. C° is nonempty
- ii. C contains a basis of M .
- iii. $\langle C \rangle = M$

Proposition : Let C be a nonempty closed convex cone in M . Then the following properties are equivalent :

- i. C is pointed
- ii. C is proper

iii. $\text{int}(C^*) \neq \phi$

As a consequence, we have

Corollary : Let C be a closed convex cone. Then C is proper if and only if C^* is generating.

Corollary : Let C be a convex cone in M . Then $C \in \text{Cone}(M)$ if and only if $C^* \in \text{Cone}(M)$.

Here $\text{Cone}(M)$ is the set of all closed regular convex cones in M .

To proceed further along these lines, we need to make ourselves familiar with more terminology and notations. The linear automorphism group of a convex cone is defined as follows:

$\text{Aut}(C) = \{a \in \text{GL}(M) / \alpha(C) = C\}$. $\text{GL}(M)$ is the group of invertible linear transformations of M . If C is open or closed, $\text{Aut}(C)$ is closed in $\text{GL}(M)$. In particular $\text{Aut}(C)$ is a linear Lie group.

Definition : Let G be a group acting linearly on M . Then a cone $C \in M$ is called G -invariant if $G.C = C$. We denote the set of invariant regular cones in M by $\text{Cone}_G(M)$. A convex cone C is called *homogeneous* if $\text{Aut}(C)$ acts transitively on C .

For $C \in \text{Cone}_G(M)$, we have $\text{Aut}(C) = \text{Aut}(C^o)$ and $C = \partial C \cup C^o = (C - C^o) \cup C^o$ is a decomposition of C into $\text{Aut}(C)$ -invariant subsets. In particular a non-trivial closed *regular* cone can never be homogeneous. We now state the following theorem:

Theorem : Let G be a Lie group acting linearly on the Euclidean vector space M and $C \in \text{Cone}_G(M)$. Then the stabilizer in G of a point in C^o is compact.

In the abstract mathematical setting, cones are described using quasi-order relation [24] as follows: Let $M \neq 0$ be a set and $*$ be a mapping of $M \times M$ into $P^*(M)$ (the set of all non-empty subsets of M). The pair $(M, *)$ is called a hypergroupoid. For $A, B \in P^*(M)$, we define $A * B = \bigcup \{a * b : a \in A, b \in B\}$.

A hypergroupoid $(M, *)$ is called a hypergroup, if $(a * b) * c = a * (b * c)$ for all $a, b, c \in M$, and the reproduction axiom, $a * M = M = M * a$, for any $a \in M$, is satisfied.

For a binary relation R on A and $a \in A$ denote $U_R(a) = \{b \in A / \langle a, b \rangle \in R\}$. A binary relation Q on a set A is called quasiorder if it is reflexive and transitive. The set $U_Q(a)$ is called a cone of a . In the case when a quasiorder Q is an equivalence, $U_Q(A) = \{x \in M / \exists y \in A, \langle x, y \rangle \in Q\}$ for any $A \subseteq M$. Analogously, for $B \subseteq A$ we set $U_Q(B) = \bigcap \{U_Q(a) / a \in B\}$.

In the light of this definition, we shall observe in section 3 that causal cones and K-causal cones fall in this category since causal relation $<$ and K-causal relation \prec are reflexive and transitive.

In the literature, (see for example [25, 26, 27]), cone preserving mappings are defined as follows:

Let $\mathbf{A} = (A, R)$ and $\mathbf{B} = (B, S)$ be quasi-ordered sets. A mapping $h : A \rightarrow B$ is called *cone preserving* if $h(U_R(a)) = U_S(h(a))$ for each $a \in A$.

To illustrate the concepts described above, we consider the following example:

Example of a Forward Light cone in Minkowski space :

Note: In the paper by Gheorghie and Mihul [28], forward light cone is called ‘*positive cone*’ and is defined as follows:

Let M be a n -dimensional real linear space. A causal relation of M is a partial ordering relation \geq of M with regard to which M is *directed*, i.e. for any $x, y \in M$ there is $z \in M$ so that $z \geq x, z \geq y$. Then the positive cone is defined as $C = \{x / x \in M; x \geq 0\}$

Let p and q be two positive integers and $n = p + q$. Let $M = R^n$. We write elements of M as $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with $x \in R^p$ and $y \in R^q$. For $p = 1$, x is a real number.

We write projections p_{r_1} and p_{r_2} as $p_{r_1}(v) = x$ and $p_{r_2}(v) = y$.

As discussed earlier, connected component of identity in $O(p, q)$ denoted by $O(p, q)_o = SO_0(p, q) = SO(p, q)_o$. Also Let

$Q_{+r} = \{x \in R^{n+1} / Q_{p+1, q}(x, x) = r^2\}, r \in R^+, p, q \in N, n = p + q \geq 1$.

Clearly, $O(p+1, q)$ acts on Q_{+r} . Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis for R^n . Then we have the following result.

Proposition : For $p, q > 0$, the group $SO_0(p+1, q)$ acts transitively on Q_{+r} . The isotropy subgroup at re_1 is isomorphic to $SO_0(p, q)$. As a manifold,

$Q_{+r} \simeq SO_0(p+1, q)/SO_0(p, q)$.

In particular for $n \geq 2, q = n - 1$ and $p = 1$, we define the semi algebraic cone C in R^n by

$C = \{v \in R^n / Q_{1,q}(v, v) \geq 0, x \geq 0\}$ and set

$C^* = \Omega = \{v \in R^n / Q_{1,q}(v, v) > 0, x > 0\}$. C is called the *forward light cone* in R^n . We have

$M = \begin{pmatrix} x \\ y \end{pmatrix} \in C$ if and only if $x \geq \|y\|$.

(Gheorghe and Mihul [28] state in Lemma 1 that *There is a norm $\|\cdot\|$ in \overline{M} (a $n-1$ dimensional linear real space) so that: $Q = \{x/x \in M; \varepsilon x^0 = \|\overline{x}\|\}$, $intC = \{x/x \in M; \varepsilon x^0 > \|\overline{x}\|\}$, where $\varepsilon = 1$ if $(-1, \overline{0})$ is not in C and $\varepsilon = -1$ if $(1, \overline{0})$ is not in C .*

Boundary of C and C^o are described as follows: $\partial C = \{v \in R^n / \varepsilon x = \|y\|\}$, $C^o = \{v \in R^n / \varepsilon x > \|y\|\}$ where $\varepsilon = 1$ if $(-1, 0)$ is not in C and $\varepsilon \neq 1$ if $(1, 0) \in C$.

If $v \in C \cap -C$, then $0 \leq x \leq 0$ and hence $x = 0$. Then $\|y\| = 0$ and thus $y = 0$. Thus $v = 0$ and C is proper.

For $v, v' \in C$, we calculate

$(v, v') = (v', v) = x'x + (y', y) \geq \|y'\| \|y\| + (y', y) \geq 0$. Thus $C \subset C^*$.

Conversely, let $v = \begin{pmatrix} x \\ y \end{pmatrix} \in C^*$. Then testing against e_1 , we get $x \geq 0$. We may assume $y \neq 0$.

Define ω by $p_{r_1}(\omega) = \|y\|$ and $p_{r_2}(\omega) = -y$. Then $\omega \in C$ and $0 \leq (w, v) = x \|y\| - \|y\|^2 = (x - \|y\|) \|y\|$. Hence $x \geq \|y\|$. Therefore $y \in C$ and thus $C^* \subset C$. So $C = C^*$ and C is self-dual. Similarly, we can show that Ω is self dual.

Moreover, the forward light cone C is invariant under the usual operation of $SO_o(1, q)$ and under all dilations, $\lambda I_n, \lambda > 0$. (I_n is the $n \times n$ identity matrix). We now prove that the group $SO_o(1, q)R^+I_{q+1}$ acts transitively on $\Omega = C^o$ if $q \geq 2$ ($q = 3$ for Minkowski space). Thus Ω will

be homogeneous. For this we prove that $\Omega = SO_o(1, q)R^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Using

$a_t = \begin{pmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \in SO_o(1, q)$, we get

$a_t \begin{pmatrix} \lambda \\ 0 \end{pmatrix} = \lambda^t (\cosh(t), \sinh(t), 0, \dots, 0)$ for all $t \in R$. Let S^{q-1} denote a unit sphere in R^q . Now

$SO(q)$ acts transitively on S^{q-1} and $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in SO_o(1, q)$

for all $A \in SO(q)$. Hence the result follows by noting the fact that $\coth(t)$ runs through $(1, \infty)$ as t varies in $(0, \infty)$.

There is a vast literature on homogeneous convex cones and they are used in convex optimization problems. See, for example, the paper by Truong and Tunçel [13] and references therein.

3 Causal Structure of Space-times, Causality Conditions and Causal group

In this section, we begin with basic definitions and properties of causal structure of space-time. Then we define different causality conditions and their hierarchy. Furthermore we discuss causal group and causal topology on space-time in general, and treat Minkowski space as a special case. We take a space-time (M, g) as a connected C^2 -Hausdorff four dimensional differentiable manifold which is paracompact and admits a Lorentzian metric g of signature $(-, +, +, +)$. Moreover, we assume that the space-time is space and time oriented.

We say that an event x *chronologically* precedes another event y , denoted by $x \ll y$ if there is a smooth future directed timelike curve from x to y . If such a curve is non-spacelike, i.e., timelike or null, we say that x causally precedes y or $x < y$. The chronological future $I^+(x)$ of x is the

set of all points y such that $x \ll y$. The chronological past $I^-(x)$ of x is defined dually. Thus

$$I^+(x) = \{y \in M/x \ll y\} \text{ and}$$

$$I^-(x) = \{y \in M/y \ll x\}.$$

The causal future and causal past for x are defined similarly :

$$J^+(x) = \{y \in M/x < y\} \text{ and}$$

$$J^-(x) = \{y \in M/y < x\}$$

As Penrose [29] has proved, the relations \ll and $<$ are transitive. Moreover, $x \ll y$ and $y < z$ or $x < y$ and $y \ll z$ implies $x \ll z$. Thus $\overline{I^+(x)} = \overline{J^+(x)}$ and also $\partial I^+(x) = \partial J^+(x)$, where for a set $X \subset M$, \overline{X} denotes closure of X and ∂X denotes topological boundary of X . The chronological future and causal future of any set $X \subset M$ is defined as

$$I^+(X) = \bigcup_{x \in X} I^+(x) \text{ and}$$

$$J^+(X) = \bigcup_{x \in X} J^+(x)$$

The chronological and causal pasts for subsets of M are defined similarly.

An ordering which is reflexive and transitive is called quasi - ordering. This ordering was developed in a generalized sense by Sorkin and Woolgar [7] and these concepts were further developed by Garcia Parrado and Senovilla [30, 31] and S. Janardhan and Saraykar [8] to prove many interesting results in causal structure theory in GR.

In the recent paper, Zapata and Kreinovich [27] call chronological order as open order and causal order as closed order and prove that under reasonable assumptions, one can uniquely reconstruct an open order if one knows the corresponding closed order. For special theory of relativity, this part is true and hence every one-one transformation preserving a closed order preserves open order and topology. This fact in turn implies that every order preserving transformation is linear. The converse part is well known namely, the open relation uniquely determines both the topology and the closed order.

We now introduce the concept of K-causality and give causal properties of space-times in the light of this concept. For more details we refer the reader to [8], [10, 11] and [30, 31].

Definition : K^+ is the smallest relation containing I^+ that is topologically closed and transitive. If q is in $K^+(p)$ then we write $p \prec q$.

That is, we define the relation K^+ , regarded as a subset of $M \times M$, to be the intersection of all closed subsets $R \supseteq I^+$ with the property that $(p, q) \in R$ and $(q, r) \in R$ implies $(p, r) \in R$. (Such sets R exist because $M \times M$ is one of them.) One can also describe K^+ as the closed-transitive relation generated by I^+ .

Definition : An open set O is *K-causal* iff the relation ' \prec ' induces a reflexive partial ordering on O . i.e. $p \prec q$ and $q \prec p$ together imply $p = q$.

If we regard C^o as the interior of future light cone in a Minkowski space-time ($p = 1, q = 3$), then under standard chronological structure $I^+, M(a, b)$ becomes $I^-(b) \cap I^+(a)$. As it is well known, such sets form a base for Alexandrov topology and since Minkowski space-time is globally hyperbolic and hence strongly causal, Alexandrov topology coincides with the manifold topology (Euclidean topology). Thus, lemma 2 of [28] is a familiar result in the language of Causal structure theory.

Analogous to usual causal structure, we defined in [8] strongly causal and future distinguishing space-times with respect to K^+ relation.

Definition : A C^0 - space-time M is said to be *strongly causal at p with respect to K^+* , if p has arbitrarily small K - convex open neighbourhoods.

Analogous definition would follow for K^- .

M is said to be *strongly causal with respect to K^+* , if it is strongly causal with respect to K^+ at each and every point of it. Thus, lemma 16 of [7] implies that K-causality implies strong causality with respect to K^+ .

Definition : A C^0 - space-time M is said to be *K-future distinguishing* if for every $p \neq q, K^+(p) \neq K^+(q)$. *K-past distinguishing* spaces can be defined analogously.

Definition : A C^0 - space-time M is said to be *K-distinguishing* if it is both K-future and K-past distinguishing.

Analogous result would follow for K^- . Hence, in a C^0 - space-time M , strong causality with respect to K implies K-distinguishing.

Remark : K-conformal maps preserve K- distinguishing, strongly causal with respect to K^+ and globally hyperbolic properties.

Definition : A C^0 - space-time M is said to be *K-reflecting* if

$$K^+(p) \supseteq K^+(q) \Leftrightarrow K^-(q) \supseteq K^-(p).$$

However, since the condition $K^+(p) \supseteq K^+(q)$ always implies $K^-(q) \supseteq K^-(p)$ because of transitivity and $x \in K^+(x)$, and vice versa, a C^0 - space-time with K-causal condition is always K-reflecting. Moreover, in general, K-reflecting need not imply reflecting. Since, any K-causal space-time is K-reflecting, any non-reflecting open subset of the space-time will be K-causal but non-reflecting.

We now give the interesting hierarchy of K-causality conditions as follows:

We have proved that strong causality with respect to K^+ implies K-future distinguishing. Thus, K-causality \Rightarrow strongly causality with respect to K \Rightarrow K - distinguishing.

Since a K- causal space-time is always K-reflecting, it follows that the K-causal space-time is K-reflecting as well as K-distinguishing. In the classical causal theory, such a space-time is called causally continuous [32]. (Such space-times have been useful in the study of topology change in quantum gravity [33]). Thus if we define K-causally continuous space-time analogously then we get the result that a K-causal C^0 - space-time is K-causally continuous. Moreover, since $K^\pm(x)$ are topologically closed by definition, analogue of causal simplicity is redundant and causal continuity (which is implied by causal simplicity) follows from K-causality. In [10], E. Minguzzi proved the equivalence of K-causality and stable causality.

Thus the causal hierarchy reads as follows.

$$\text{Global hyperbolicity} \Rightarrow \text{Stably causal} \Leftrightarrow \text{K-causality} \Rightarrow \text{Strong causality} \\ \Rightarrow \text{K - Distinguishing.}$$

We now proceed to discuss causal groups and causal topology. We then compare these notions with those in section 2.

If R^n is a directed set with respect to a certain partial ordering relation ' \geq ' of R^n , then such a relation is called a *Causal relation*. Thus in a globally hyperbolic space-time (or in a Minkowski space time) J^+ and K^+ are causal relations (In a C^2 globally hyperbolic space-time, $J^+ = K^+$, whereas in a C^0 - globally hyperbolic space-time, only K^+ is valid). The *Causal group* G relative to causal relation is then defined as the group of permutations $f : R^n \rightarrow R^n$ which leaves the relation ' \geq ' invariant. i.e. $f(x) \geq f(y)$ if and only if $x \geq y$. Such maps are called causal maps. They preserve causal order. These maps are special cases of cone preserving maps defined in section 2.

Thus in a C^0 globally hyperbolic space-time, every K - causal map f where f^{-1} is also order preserving is a causal relation and causal group is the group of all such mapping which we called K - conformal groups.

In the light of the definition of quasiorder given in section 2, we observe that causal cones and K - causal cones fall in this category, since causal relation ' $<$ ' and K - causal relation ' \prec ' are reflexive and transitive. If we replace quasi-order by a causal relation (or K-causal relation), then we see that an order preserving map is nothing but a causal map. Thus an order preserving map is a generalization of a causal map (or K-causal map). These concepts also appear in a branch of theoretical computer science called domain theory. Martin and Panangaden [12] and S. Janardhan and Saraykar [9] have used these concepts in an abstract setting and proved some interesting results in causal structure of space times. They proved that order gives rise to a topological structure.

As far as the *causal topology* on R^n is concerned, it is defined as the topology generated by the

fundamental system of neighbourhoods containing open ordered sets

$M(a, b)$ defined for any $a, b \in R^n$ with $b - a \in C^o$ as : $M(a, b) = \{y \in R^n / b - y, y - a \in C^o\}$. Gheorghe and Mihul [28] describe ‘causal topology’ on R^n and prove that the causal topology of R^n is equivalent to the Euclidean topology. Causal group is thus comparable to conformal group of space-time under consideration. Further any $f \in G$ is a homeomorphism in causal topology and hence it is a homeomorphism in Euclidean topology.

If C is a Minkowski cone as discussed in the above example, then Zeeman [34] has proved that G is generated by translations, dilations and orthochronous Lorentz transformations of Minkowski space R^n ($n = 4$).

We can say more for the causal group G of Minkowski space.

Let $G_0 = \{f \in G / f(0) = 0\}$.

Then G_0 contains the identity homeomorphism. Gheorghe and Mihul [28] proved that G is generated by the translations of R^n and by linear transformation belonging to G_0 . Hence G is a subgroup of the affine group of R^n . This is the main result of [28].

Let $G'_0 = G_0 \cap SL(n, R)$. Then G'_0 is the orthochronous Lorentz group under the norm $\|y\| = \left[\sum_{i=1}^q |y^i|^2 \right]^{\frac{1}{2}}$ for $y \in R^q, y = (y^1, y^2, \dots, y^q)$.

For $\|y\| = \left[\sum_{i=1}^q |y^i|^\alpha \right]^{\frac{1}{\alpha}}, \alpha > 2$, G'_0 is the discrete group of permutations and the symmetries

relative to the origin of the basis vectors of R^q . The factor group G_0/G'_0 is the dilation group of R^n . Also, G is the semi-direct product of the translation group with the subgroup G'_0 of $SL(n, R)$. Moreover G'_0 is a topological subgroup of $SL(n, R)$. Similar results have been proved by Borchers and Hegerfeldt [35]. Thus we have,

Theorem : Let M denote n -dimensional Minkowski space, $n \geq 3$ and let T be a 1 - 1 map of M onto M . Then T and T^{-1} preserve the relation $(x - y)^2 > 0$ if and only if they preserve the relation $(x - y)^2 = 0$. The group of all such maps is generated by

- i) The full Lorentz group (including time reversal)
- ii) Translations of M
- iii) Dilations (multiplication by a scalar)

In our terminology, T is a causal map.

In the same paper [35], the following theorem is also proved.

Theorem : Let $\dim M \geq 3$, and let T be a 1 - 1 map of M onto M , which maps light like lines onto (arbitrary) straight lines. Then T is linear.

This implies that constancy of light velocity c alone implies the Poincare group upto dilations.

Thus, for Minkowski space, things are much simpler. For a space-time of general relativity (a Lorentz manifold) these notions take a more complicated form where partial orders are J^+ or K^+ .

4 Causal Orientations and order theoretic approach to Global Hyperbolicity

In this section, we discuss briefly the concepts of *Causal Orientations*, *Causal Structures* and *Causal Intervals* which lead to the definition of a ‘*Globally hyperbolic homogeneous space*’.

These notions cover Minkowski Space and homogeneous cosmological models in general relativity. We also discuss domain theoretic approach to causal structure of space-time and comment on the parallel concepts appearing in these approaches.

Let M be a C^1 (respectively smooth) space-time. For $m \in M, T_m(M)$ denotes the tangent space of M at m , and $T(M)$ denotes the tangent bundle of M . The derivative of a differentiate map $f : M \rightarrow N$ at m will be denoted by $d_m f : T_m M \rightarrow T_{f(m)} N$. A C^1 (respectively smooth) causal

structure on M is a map which assigns to each point $m \in M$ a nontrivial closed convex cone $C(m)$ in $T_m M$ and it is C^1 (smooth) in the following sense:

We can find an open covering

$\{U_i\}_{i \in I}$ of M , smooth maps $\phi_i : U_i \times R^n \rightarrow T(M)$ with $\phi_i(m, M) \in T_m(M)$ and a cone C in R^n such that $C(m) = \phi_i(m, C)$.

The causal structure is called *generating* (respectively proper, regular) if $C(m)$ is generating (proper, regular) for all m . A map $f : M \rightarrow M$ is called *causal* if $d_m f(C(m)) \subset C(f(m))$ for all $m \in M$. These definitions are obeyed by causal structure J^+ in a causally simple space-time and causal maps of García-Parrado and Senovilla [31]. If we consider C^0 -Lorentzian manifold with a C^1 -metric so that we can define null cones, then these definitions are also satisfied by causal structure K^+ and K -causal maps. Thus the notions defined above are more general than those occurring in general relativity at least in a special class of space-times. Rainer [36] called such a causal structure an *ultra weak cone structure* on M where $m \in \text{int}M$.

We now define G -invariant causal structures where G is a Lie group and discuss some properties of such structures. If a Lie group G acts smoothly on M via $(g, m) \mapsto g.m$, we denote the diffeomorphism $m \mapsto g.m$ by l_g .

Definition: Let M be a manifold with a causal structure and G a Lie group acting on M . Then the causal structure is called G -invariant if all l_g , $g \in G$, are causal maps. If H is a Lie subgroup of G and $M = G/H$ is homogeneous then a G -invariant causal structure is determined completely by the cone $C = C(0) \subset T_o M$, where $o = H \in G/H$. Moreover C is proper, generating etc if and only if this holds for the causal structure. We also note that C is invariant under the action of H on $T_o(M)$ given by $h \mapsto d_0 l_h$. On the other hand, if $C \in \text{Cone}_H(T_o(M))$, then we can define a field of cones by $M \rightarrow T_{\alpha.o}(M)$:

$$\alpha H \mapsto C(\alpha H) = d_0 l_\alpha(C).$$

This cone field is G -invariant, regular and satisfies $C(0) = C$. Moreover the mapping $m \mapsto C(m)$ is also smooth in the sense described above. If this mapping is only continuous in the topological sense, for all m in M , then Rainer [36] calls such cone structure, a *weak local cone structure* on M . We have the following theorem.

Theorem : Let $M = G/H$ be homogeneous. Then $C \mapsto (\alpha H \mapsto d_0 l_\alpha(C))$ defines a bijection between $\text{Cone}_H(T_o(M))$ and the set of G -invariant, regular causal structures on M .

We call a mapping $\nu : [a, b] \rightarrow M$ as *absolutely continuous* if for any coordinate chart $\phi : U \rightarrow R^n$, the curve $\eta = \phi \circ \nu : \nu^{-1}(U) \rightarrow R^n$ has absolutely continuous coordinate functions and the derivatives of these functions are locally bounded.

Further, we define a C -causal curve: Let $M = G/H$ and $C \in \text{Cone}_G(T_o M)$. An absolutely continuous curve $\nu : [a, b] \rightarrow M$ is called C -causal (*Cone causal or conal*) if $\nu'(t) \in C(\nu(t))$ whenever the derivative exists.

Next, we define a relation ' \leq_s ' (s for strict) of M by $m \leq_s n$ if there exists a C -causal curve ν connecting m with n . This relation is obviously reflexive and transitive. Such relations are called *causal orientations* or *quasi-orders*. They give rise to causal cones as we saw in section 2.

Note : A reader who is familiar with the books by Penrose [29], Hawking and Ellis [3] or Joshi [5] will immediately note that the above relation is our familiar causal order J^\pm in the case when M is a space-time in general relativity.

We ask the question : Which of the space-times M can be written as G/H ? Gödel universe, Taub universe and Bianchi universe are some examples of such space-times. They are all spatially homogeneous cosmological models. Isometry group of a spatially homogeneous cosmological model may or may not be abelian. If it is abelian, then these are of Bianchi type I, under Bianchi classification of homogeneous cosmological models. Thus above discussion applies to such models.

As an example to illustrate above ideas, we again consider a finite dimensional vector space M and let C be a closed convex cone in M . Then we define a causal $\text{Aut}(C)$ -invariant orientation on M by $u \leq v$ iff $v - u \in C$. Then ' \leq ' is antisymmetric iff C is proper. In particular $H^+(n, R)$ defines a $GL(n, R)$ -invariant global ordering in $H(n, R)$. Here $H(n, R)$ are $n \times n$ real orthogonal

matrices (Hermitian if R is replaced by C) and $H^+(n, R) = \{X \in H(n, R) / X \text{ is positive definite}\}$ is an open convex cone in $H(n, R)$. (the closure of $H^+(n, R)$ is the closed convex cone of all positive semi definite matrices in $H(n, R)$). Also, the light cone $C \subset R^{n+1}$ defines a $SO_O(n, 1)$ -invariant ordering in R^{n+1} . The space R^{n+1} together with this global ordering is the $(n+1)$ -dimensional Minkowski space.

Going back to the general situation we note that in general, the graph

$M_{\leq_s} = \{(m, n) \in M \times M / m \leq_s n\}$ of ' \leq_s ' is not closed in $M \times M$. However, if we define $m \leq n \Leftrightarrow (m, n) \in \overline{M_{\leq_s}}$, then it turns out that ' \leq ' is a causal orientation. This can be seen as follows:

' \leq ' is obviously reflexive. We show that it is transitive:

Suppose $m \leq n \leq p$ and let m_k, n_k, n'_k, p_k be sequences such that $m_k \leq_s n_k$, $n'_k \leq_s p_k$, $m_k \rightarrow m$, $n_k \rightarrow n$, $n'_k \rightarrow n$ and $p_k \rightarrow p$. Now we can find a sequence g_k in G converging to the identity such that $n'_k = g_k n_k$. Thus $g_k m_k \rightarrow m$ and $g_k n_k \leq_s p_k$ implies $m \leq p$. The above result resembles the way in which K^+ was constructed from I^+ .

The following definitions are analogous to I^\pm, J^\pm or K^\pm and so is the definition of interval as $I^+(p) \cap I^-(q) (J^+(p) \cap J^-(q)$ or $K^+(p) \cap K^-(q)$):

Given any causal orientation ' \leq ' on M , we define for $A \subset M$,

$\uparrow A = \{y \in M / \exists a \in A \text{ with } a \leq y\}$ and

$\downarrow A = \{y \in M / \exists a \in A \text{ with } y \leq a\}$.

Also, we write $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$.

The intervals with respect to this causal orientation are defined as

$[m, n]_{\leq} = \{z \in M / m \leq z \leq n\} = \uparrow m \cap \downarrow n$.

Finally we introduce some more definitions.

Definitions : Let M be a space-time.

- (1) a causal orientation ' \leq ' on M is called *topological* if its graph M_{\leq} in $M \times M$ is closed.
- (2) a space (M, \leq) with a topological causal orientation is called a *causal space*. If ' \leq ' is, in addition, antisymmetric, that is a partial order, then (M, \leq) is called *globally ordered* or *ordered*.
- (3) Let (M, \leq) and (N, \leq) be two causal spaces and let $f : M \rightarrow N$ be continuous. Then f is called *order preserving* or *monotone* if $m_1 \leq m_2 \Rightarrow f(m_1) \leq f(m_2)$.
- (4) Let G be a group acting on M . Then a causal orientation \leq is called *G-invariant* if $m \leq n \Rightarrow a.m \leq a.n, \forall a \in G$.
- (5) A triple (M, \leq, G) is called a *Causal G-Manifold* or *causal* if ' \leq ' is a topological G -invariant causal orientation.

Thus referring to partial order K^+ , we see, in the light of above definitions (1) and (2), that \leq_K is topological and (M, \leq_K) is a causal space. A K -causal map satisfies definition (3).

For a homogeneous space $M = G/H$ carrying a causal orientation such that (M, \leq, G) is causal, the intervals are always closed subsets of M . If the intervals are compact, we say that $M = G/H$ is *globally hyperbolic*. We use the same definition for a space-time where intervals are $J^+(p) \cap J^-(q)$. Thus globally hyperbolic space-times can be defined by using causal orientations for homogeneous spaces. In this setting, intervals are always closed, as in causally continuous space-times.

As the last part of our review, we discuss the central concepts and definitions of domain theory, as we observe that these concepts are related to causal structure of space-time and also to space-time topologies.

The relations $<$ and \ll discussed in section 3 have been generalised to abstract orderings using the concepts in Domain Theory and also many interesting results have been proved related to causal structures of space - time in general relativity. For definitions and preliminary results in domain theory, we follow Abramsky and Jung [37] and Martin and Panangaden [12].

A *poset* is defined as a partially ordered set, i.e. a set together with a reflexive, anti-symmetric and transitive relation. A concept that plays an important role in the theory is the one of a directed subset of a domain, i.e. of a non-empty subset in which each two elements have an upper bound.

Definition : Let (P, \sqsubseteq) be a partially ordered set. An *upper bound* of a subset S of a poset P

is an element b of P , such that $x \sqsubseteq b, \forall x \in S$. The dual notion is called *lower bound*.

Definition : A nonempty subset $S \subseteq P$ is *directed* if for every x, y in $S, \exists z \in S \ni: x, y \sqsubseteq z$. The supremum of $S \subseteq P$ is the least of all its upper bounds provided it exists and is denoted by $\sqcup S$.

A nonempty subset $S \subseteq P$ is *filtered* if for every x, y in $S, \exists z \in S \ni: z \sqsubseteq x, y$. The infimum of $S \subseteq P$ is the greatest of all its lower bounds provided it exists and is denoted by $\bigwedge S$.

Definition : A *dcpo* is a poset in which every directed subset has a supremum.

Using partial order some topologies can be derived. For example,

Definition : A subset U of a poset P is *Scott open* if

- (i) U is an upper set: i.e. $x \in U$ and $x \sqsubseteq y \Rightarrow y \in U$ and
- (ii) U is inaccessible by directed suprema: i.e. for every directed $S \subseteq P$ with a supremum, $\sqcup S \in U \Rightarrow S \cap U \neq \emptyset$.

The collection of all Scott open sets on P is called the *Scott topology*.

The order of *approximation* denoted by ' \ll ' is defined as:

Definition : For elements x, y of a poset, $x \ll y$ iff for all directed sets S with a supremum, $y \sqsubseteq \sqcup S \Rightarrow \exists s \in S \ni: x \sqsubseteq s$.

Define, $\downarrow x = \{a \in D / a \ll x\}$ and $\uparrow x = \{a \in D / x \ll a\}$.

The special property of the finite elements x is that they are way-below themselves, i.e. $x \ll x$.

An element with this property is also called *compact*.

Definition : For a subset X of a poset P , define

$\uparrow X := \{y \in P / \exists x \in X, x \sqsubseteq y\}$ and

$\downarrow X := \{y \in P / \exists x \in X, y \sqsubseteq x\}$

Then, $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$ for $x \in X$.

Definition : A *basis* for a poset D is a subset B such that $B \cap \downarrow x$ contains a directed set with supremum x for all x in D . A poset is continuous if it has a basis. A poset is ω -continuous if it has a countable basis.

Then we have,

Theorem: The collection $\{\uparrow x / x \in D\}$ is a basis for the Scott topology on a continuous poset.

Lawson topology is defined as,

Definition : The *Lawson topology* on a continuous poset P has as a basis all sets of the form $\uparrow x \sim \uparrow F$, for $F \subseteq P$ finite.

Definition : A continuous poset P is *bicontinuous* if for all x, y in P

$x \ll y$ iff for all filtered $S \subseteq P$ with an infimum, $\bigwedge S \sqsubseteq x \Rightarrow \exists s \in S \ni: s \sqsubseteq y$ and for each $x \in P$, the set $\uparrow x$ is filtered with infimum x .

Theorem: On a bicontinuous poset P , sets of the form $(a, b) := \{x \in P / a \ll x \ll b\}$ form a basis for a topology. This topology is called the *interval topology*.

Definition : The Alexandrov topology on a space-time has $\{I^+(p) \cap I^-(q) / p, q \in M\}$ as a basis. For a pre-ordered set P , any upper set O is Alexandrov-open. Inversely, a topology is Alexandrov if any intersection of open sets is open.

Let $I^\pm(p)$ and $J^\pm(p)$ be defined as in section 3. The relation J^+ can be defined

as $p \sqsubseteq q \equiv q \in J^+(p)$. Then we have the following:

Proposition : Let $p, q, r \in M$. Then

(i) The sets $I^+(p)$ and $I^-(p)$ are open.

(ii) $p \sqsubseteq q$ and $r \in I^+(q) \Rightarrow r \in I^+(p)$

(iii) $q \in I^+(p)$ and $q \sqsubseteq r \Rightarrow r \in I^+(p)$

(iv) $I^+(p) = J^+(p)$ and $I^-(p) = J^-(p)$.

Theorem : A space-time M is strongly causal iff its Alexandrov topology is Hausdorff iff its Alexandrov topology is the manifold topology.

Proposition : $x \ll y \Leftrightarrow y \in I^+(x)$ for all x, y in M .

Theorem : (M, \sqsubseteq) is a bicontinuous poset with $\ll = I^+$ whose interval topology is the manifold topology.

Causal simplicity also has a characterization in order-theoretic terms.

Theorem : Let (M, \sqsubseteq) be a continuous poset with $\sqsubseteq = I^+$. Then the following are equivalent:

- (i) M is causally simple.
- (ii) The Lawson topology on M is a subset of the interval topology on M .

Definition : A *globally hyperbolic* space-time (M, \sqsubseteq) is a bicontinuous poset whose intervals are compact in the interval topology on M .

Bicontinuity ensures that the topology of M , that is, the interval topology is implicit in \sqsubseteq .

Theorem: A globally hyperbolic poset is locally compact Hausdorff. Also,

- (i) The Lawson topology is contained in the interval topology.
- (ii) Its partial order \sqsubseteq is a closed subset of M^2 .

We extended and generalized in [9], some of the above concepts to K -causality in a C^0 -globally hyperbolic space-time as follows.

Result: In a C^0 -globally hyperbolic space-times, $x \ll y \Rightarrow y \in K^+(x)$ where the partial order is $\ll = K^+$.

It must be noted that above analysis does not require any kind of differentiability conditions on a space-time manifold, and results are purely topological and order-theoretic.

We illustrate, for Lawson topology, as to how the concepts above can be generalized to K -causality.

We also have analogous to above,

$$\downarrow x = \{a \in M / a \ll x\} \text{ and}$$

$$\uparrow x = \{a \in M / x \ll a\} .$$

Since $a \ll x \Rightarrow a \in K^-(x)$, we have,

$$\downarrow x \subseteq K^-(x) \text{ and } \uparrow x \subseteq K^+(x).$$

Let us now take a basis for Lawson topology as the sets of the form

$$\{\uparrow x \sim \uparrow F / F \text{ is finite } \}.$$

Since F is finite, F is compact in the manifold topology and hence $\uparrow F$ is closed. Since the sets $\downarrow x$ and $\uparrow x$ are open in the manifold topology (in a C^0 -globally hyperbolic space-time), $\uparrow x \sim \uparrow F$ are also open in the manifold topology .

Thus Lawson open sets are open in the manifold topology also and hence Lawson topology, in K -sense, is contained in the manifold topology.

Similar analysis can be given for Scott topology and interval topology also. The intervals defined above, with appropriate cone structure coincide with causal intervals and hence so does the definition of global hyperbolicity. When the partial order is J^+ , interval topology coincides with Alexandrov topology and as is well-known, in a strongly causal space-time, Alexandrov topology coincides with the manifold topology.

5 Concluding Remarks

We note that there are a large number of space-times (solutions of Einstein field equations) which are inhomogeneous (see Krasinski [38]) and hence do not fall in the above class : $M = G/H$.

M.Rainer [36] defines yet another partial order using cones as subsets of a topological manifold and a differential manifold (space-time) which is a causal relation in the sense defined above and which is more general than J^+ . Rainer, furthermore defines analogous causal hierarchy like in the classical causal structure theory. Of course, for Minkowski space, the old and new definitions coincide. For a C^2 -globally hyperbolic space-time J^+, K^+ and Rainer's relation all coincide, whereas for a C^0 -globally hyperbolic space-time, K^+ and Rainer's relation on topological manifold coincide.

Moreover if the cones are characteristic surfaces of the Lorentzian metric, then all his definitions of causal hierarchy coincide with the classical definitions. (cf theorem 2 of Rainer [36]). For more details on this partial order, we refer the reader to this paper.

B.Carter [39] discusses causal relations from a different perspective and discusses in detail many features of this relationship. Topological considerations in the light of time-ordering have been discussed by E.H.Kronheimer [40].

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