arXiv:gr-qc/0510005v3 13 Nov 2009

The Kasner Brane.

Mark D. Roberts,
Max Planck Institute for Gravitational Physics (Albert Einstein Institute)
1 Am Mühlenberg, Golm, Germany D-14476
mdr@ihes.fr

12^{th} of November 2009

Abstract

Solutions are found to field equations constructed from the Pauli, Bach and Gauss-Bonnet quadratic tensors to the Kasner and Kasner brane spacetimes in up to five dimensions. A double Kasner space is shown to have a vacuum solution. Brane solutions in which the bulk components of the Einstein tensor vanish are also looked at and for four branes a solution similar to radiation Robertson-Walker spacetime is found. Matter trapping of a test scalar field and a test perfect fluid are investigated using energy conditions.

gr-qc/0510005

Contents

1 Introduction

Randall and Sundrum [9] considered models where four dimensional spacetime is multiplied by a conformal factor which has dependence on the fifth dimension. Originally the four dimensional spacetime was taken to be flat and the dependence on the fifth dimension such that overall the space was five dimensional anti-deSitter space. Apart from flat four dimensional spacetime, four dimensional Gödel spacetime Barrow and Tsagas [2] and four dimensional Bianchi type IX cosmologies van den Hoogen, Coley and He [6] have been considered. Here, instead of four dimensional flat spacetime, Kasner spacetime in various dimensions is considered, with a view to seeing what sort of field equations can be obeyed. The field equations that are looked at are those involving quadratic curvature, as such field equations often occur in fundamental theories such as string theories. There is a choice of using field equations from Lagrangians involving the Ricci scalar squared, the Ricci tensor squared, and the Riemann tensor squared, or equivalently using the Pauli, Bach and Gauss-Bonnet tensors. The later has the advantage that interpretation of the tensors is more immediate. The Pauli tensor [8] comes from varying a Ricci scalar squared Lagrangian and is defined by

$$P_{ab} = 2R_{;a;b} - 2RR_{ab} + g_{ab}(R^2/2 - 2\Box R), \tag{1}$$

as it comes from varying the simplest quadratic scalar it is the most frequantly studied. The Bach tensor [1] comes from varying the Weyl squared Lagrangian and is defined by

$$B_{ab} = 2C_{a..b}^{\ \ cd}R_{cd} + 4C_{a..b;c;d}^{\ \ cd}, \tag{2}$$

it has several unusual properties such as: trace-free, conformal invariance, asymptotically increasing (i.e. $\exp(+kr)$) spherically symmetric linearization, faster-than-light linearization. The Bach tensor often occurs in quantum theories where it cancels out divergent one-loop terms. The Gauss-Bonnet invariant [13] is

$$GB = R_{cdef}R^{cdef} - 4R_{cd}R^{cd} + R^2, \tag{3}$$

variation gives the Gauss-Bonnet tensor

$$GB_{ab} = 4R_{acde}R_b^{\ cde} - 8R_{cd}R^{c\ d}_{\ a\ b} - 8R_{ac}R^c_{\ b} + 4RR_{ab} - g_{ab}GB, \tag{4}$$

it vanishes in four dimensions, but could be the dominant low energy quadratic curvature contribution from string theory. A scalar field has

$$R_{ab} = 2\phi_a\phi_b - g_{ab}V\left(\phi^2\right), \tag{5}$$

and a perfect fluid

$$G_{ab} = (\mathcal{P}+\mu)\,U_aU_b + \mathcal{P}g_{ab},\ R_{ab} = (\mathcal{P}+\mu)\,U_aU_b + \frac{\mathcal{P}-\mu}{2-d}g_{ab}, d \neq 2. \tag{6}$$

often the equation of state

$$\mathcal{P} = (\gamma - 1)\,\mu, \tag{7}$$

is used. Calculations were done using grtensor2/maple9 [7].

2 Kasner spacetime in two dimensions.

In two dimensions the line element is

$$ds_2^2 = -dt^2 + t^{2p}dx^2, \tag{8}$$

x is a Killing coordinate, but not t. The Weyl, Bach and Gauss-Bonnet tensors either vanish or are not defined. The Ricci tensor is given by

$$R_{ab} = \frac{p(p-1)}{t^2}\begin{pmatrix} t^{2p} & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

The Pauli tensor is given by

$$P_{ab} = \frac{2p(p-1)(p+3)}{t^4}\begin{pmatrix} -(p-4)t^{2p} & 0 \\ 0 & p \end{pmatrix}, \tag{10}$$

so that when $p = -3$ there is a non-flat solution to the vacuum Pauli equations, this contrasts to the $p = 1$ case where the Ricci tensor vanishes but the spacetime is flat. The curvature invariants are

$$R = \frac{2p(p-1)}{t^2}, \quad RicciSq = \frac{1}{2}R^2, \quad RiemSq = R^2, \tag{11}$$

when $p = 1$ the spacetime is flat, otherwise there is one degree of freedom. For a vector field

$$\begin{gathered} v^a \equiv \delta^a_t, \quad norm[v] = -1, \quad v_a = [0, -1], \quad \dot{v}^a = 0, \quad \overset{v}{\Theta} = \tfrac{p}{t}, \\ \omega_{ab} = 0, \quad \sigma = \tfrac{2p}{3t}, \quad \sigma_{ab} = \delta^{xx}_{ab}\tfrac{2}{3}t^{(2p-1)}. \end{gathered} \tag{12}$$

3 Kasner spacetime in three dimensions.

The line element is

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2, \tag{13}$$

The Weyl, Bach and Gauss-Bonnet tensors either vanish or are not defined. The Ricci tensor, scalar and Pauli tensor up to symmetry in x and y are:

$$\begin{gathered} R_{xx} = (p_1 + p_2 - 1)t^{(2p_1-2)}, \quad R_{tt} = -\tfrac{1}{t^2}(-p_1 - p_2 + p_1^2 + p_2^2), \\ R = \tfrac{2}{t^2}(p_1^2 + p_2^2 + 2p_1p_2 - p_1 - p_2), \\ P_{xx} = \tfrac{R}{t^2}(p_1 - p_1^2 - p_1p_2 + p_2^2 - 5p_2 + 12), \quad P_{tt} = \tfrac{R}{t^2}(3p_1 + p_1^2 + 3p_2 + p_2^2 - p_1p_2). \end{gathered} \tag{14}$$

No solutions for this line element have been found at all. The traditional constraints $p_1 + p_2 = 1$, $p_1^2 + p_2^2 = 1$, here reduces to $p_1 = 0$, $p_2 = 1$, which gives vanishing $RiemSq$.

4 Kasner spacetime in four dimensions.

The metric is

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{15}$$

Let

$$b \equiv p_1 + p_2 + p_3, \quad a^2 \equiv p_1^2 + p_2^2 + p_3^2, \quad c \equiv p_1p_2 + p_1p_3 + p_2p_3, \tag{16}$$

then

$$2c = b^2 - a^2. \tag{17}$$

The Ricci tensor is given by

$$R_{xx} = (b-1)p_1t^{(2p_1-2)}, \quad R_{tt} = (b-a^2)t^{-2}, \tag{18}$$

where the y and z components are given by symmetry. The Bach tensor is

$$\begin{aligned} B_{xx} &= \frac{t^{p_1}}{3t^4}(2a^2-3+2b-b^2)(3a^2-b^2-4b-4p_1^2+12p_1+4p_2p_3), \\ B_{tt} &= \frac{1}{3t^4}(2a^2-3+2b-b^2)(3a^2-b^2). \end{aligned} \tag{19}$$

For the vector field

$$v^a = \delta_t^a, \tag{20}$$

the acceleration, vorticity, and magnetic part of the Weyl tensor all vanish, but the expansion is

$$\overset{v}{\Theta} = \frac{b}{t}, \tag{21}$$

the shear tensor is

$$\sigma_{xx} = -\frac{t^{2p_1-1}}{3}(-2p_1+p_2+p_3), \quad \sigma_{tt} = 0, \tag{22}$$

the shear scalar is

$$\sigma = \frac{\sqrt{3a^2-b^2}}{\sqrt{3}t}. \tag{23}$$

When $a^2 = b = 1$ both the Ricci tensor and the Bach tensor vanish, and the curvature invariants are

$$Ricciscalar = 0, \quad RicciSq = 0, \quad WeylSq = RiemSq = -\frac{16p_3^2(p_3-1)}{t^4}, \tag{24}$$

this is the traditional solution; however the Bach tensor also vanishes when

$$2a^2 - 3 + 2b - b^2 = 0 \quad \text{or} \quad b = 1 \pm \sqrt{2a^2-2}, \tag{25}$$

and this combination of constants has other solutions. For example: the *expansion-free case* when $b = 0$ & $2a^2 = 3$, 21 shows the expansion vanishes and 23 shows non-vanishing shear scalar, this has curvature products

$$R = \tfrac{3}{2t^2}, \quad RicciSq = \tfrac{15}{4t^4}, \quad WeylSq = -\tfrac{6}{t^4}(-1 - 3p_3 + 4p_3^3), \\ RiemSq = -\tfrac{3}{4t^4}(-24p_3 + 32p_3^3 - 17), \tag{26}$$

and the *shear-free case* when $a^2 = b = 3$, 23 shows there is vanishing shear scalar and 21 shows non-vanishing expansion, this has curvature products

$$R = \frac{6}{t^2}, \quad RicciSq = \frac{12}{t^4}, \quad WeylSq = 0, \quad RiemSq = \frac{12}{t^4}. \tag{27}$$

5 Kasner spacetime in five dimensions.

The line element is

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 + t^{2p_4}dw^2, \tag{28}$$

There is the traditional type solution, with

$$p_1 + p_2 + p_3 + p_4 = 1, \quad p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1, \tag{29}$$

the Ricci, Pauli and Bach tensors all vanish. The invariants are

$$RiemSq = WeylSq = -\tfrac{8}{t^4} \times \\ (-2p_3p_4 + p_3^2p_4 + p_3^3p_4 + p_4^2p_3 + p_4^3p_3 + 2p_3^3 + 2p_4^3 - 2p_3^2 - 2p_4^2 + p_3^2p4^2). \tag{30}$$

For the constraints

$$p_3 = p_4 = 0, \quad p_1 + p_2 = 1 \tag{31}$$

the Gauss-Bonnet tensor vanishes and the invariants are

$$R = \frac{2p_2(p_2 - 1)}{t^2}, \quad RiemSq = 3R^2, \quad RicciSq = R^2, \quad WeylSq = \frac{11}{6}R^2, \tag{32}$$

this is the same pattern as some scalar five dimensional solutions [10]. No other five dimensional Kasner solution has been found.

6 The Kasner one brane.

The line element is

$$ds^2 = -\exp(\frac{\chi}{\sqrt{\alpha}})dt^2 + d\chi^2, \tag{33}$$

this line element obeys the vacuum Einstein equations $G_{ab} = 0$, and has curvature invariants

$$R = -\frac{1}{2\alpha}, \quad RiemSq = R^2, \quad RicciSq = \frac{1}{2}R^2. \tag{34}$$

7 The Kasner two brane.

The line element is taken to be

$$ds^2 = \exp(\frac{\chi}{\sqrt{\alpha}})\left(-dt^2 + t^{2p}dx^2\right) + d\chi^2. \tag{35}$$

After subtracting off the cosmological constant using

$$\bar{R}_{ab} = R_{ab} + \frac{(d-1)}{4\alpha}g_{ab}, \quad \bar{G}_{ab} = G_{ab} + \frac{(d-1)(2-d)}{8\alpha}g_{ab}, \tag{36}$$

where d is the dimension of the spacetime, the curvature is given by

$$\bar{R}_{tt} = V, \quad \bar{R}_{xx} = -t^{2p}V, \quad \bar{G}_{\chi\chi} = V\exp\left(-\frac{\chi}{\sqrt{\alpha}}\right), \quad V \equiv \frac{p(p-1)}{t^2}. \tag{37}$$

The stress on the brane surface can be taken to be that of a $d = 2$ scalar field in which the derivatives of the field are negligible and the potential given by V in 37; alternatively the stress can be taken to be that of a $d = 3$ fluid with vector field $U_a = (0, 0, 1),\ \mathcal{P} = 0,\ \mu = G_{\chi\chi}$, this is less satisfactory as the fluid vector U is not timelike nor in the brane.

No solutions for the 'mixed up' brane line element

$$ds^2 = \exp(\frac{\chi}{\sqrt{\alpha}})\left(-dt^2 + t^{2p_1}dx^2\right) + t^{2p_2}d\chi^2. \tag{38}$$

have been found for $p_2 \neq 0$.

8 The Kasner three brane.

The line element is taken to be

$$ds^2 = \exp(\frac{\chi}{\sqrt{\alpha}})\left(-dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2\right) + d\chi^2. \tag{39}$$

After subtracting off the cosmological constant using 36 the modified Ricci and Einstein tensors are

$$\begin{aligned} \bar{R}_{ab} &= \frac{1}{t^2}\text{diag}\left[\sum p - \sum p^2, p_1(\sum p - 1)t^{2p_1}, p_2(\sum p - 1)t^{2p_2}, 0\right], \qquad (40) \\ \bar{G}_{ab} &= \frac{1}{t^2}\text{diag}\left[p_1p_2, p_2(1-p_2)t^{2p_1}, p_1(1-p_1)t^{2p_2}, (\sum p - \sum p^2 - p_1p_2)\exp(-\frac{\chi}{\sqrt{\alpha}})\right], \end{aligned}$$

For $p_1 = 1,\ p_2 = 0$ the line element has just the cosmological constant, for $G_{\chi\chi} = 0$ solving the quadratic gives

$$p_2 = \frac{1}{2}\left(1 - p_1 \pm \sqrt{(1-p_1)(1+3p_1)}\right), \tag{41}$$

if $p_1 = p_2$ then 41 gives both equal to 2/3 and the modified stress 41 is that of a perfect fluid 6 with

$$U_a = [1,0,0]\,, \qquad \mu = 2\mathcal{P} = \frac{4}{9t^2}, \tag{42}$$

if $p_1 \neq p_2$ then the spacetime cannot be that of a perfect fluid as a timelike fluid vector U cannot be chosen.

For $p_1 = 3,\ p_2 = 0$ the vacuum-Bach equations are obeyed and the invariants are

$$R = \tfrac{3}{\alpha t^2}\left(-t^2 + 8\alpha\exp(-\tfrac{\chi}{\sqrt{\alpha}})\right), \quad WeylSq = 192t^{-4}\exp(-\tfrac{2\chi}{\sqrt{\alpha}}),$$
$$RiemSq = \tfrac{1}{6}R^2 + \tfrac{5}{2}WeylSq, \quad RicciSq = \tfrac{1}{4}R^2 + \tfrac{3}{4}WeylSq. \tag{43}$$

9 The Kasner brane fluid.

The line element is taken to be

$$ds_5^2 = \exp\left(\frac{\chi}{\sqrt{\alpha}}\right) ds^2_{d=4\text{kasner}} + d\chi^2, \tag{44}$$

with $ds^2_{d=4\text{kasner}}$ given by 15. The metric is not Ricci-flat for any values of the p's as the $\exp(\chi/\sqrt{\alpha})$ term always occurs. After subtracting off the cosmological constant using 36 and using the notation 16 the modified Ricci and Einstein tensors are

$$\bar{R}_{ab} = \frac{1}{t^2}\left[b - a^2, t^{2p_i}p_i(b-1), 0\right], \tag{45}$$
$$\bar{G}_{ab} = \frac{1}{t^2}\left[c, b - p_i - a^2 + p_i^2, (b - a^2 - c)\exp\left(-\frac{\chi}{\sqrt{\alpha}}\right)\right],$$

where i is not summed; as for the three brane 42 take the p's all equal, the value which gives $\bar{G}_{\chi\chi} = 0$ is $p = 1/2$ and the result is a perfect fluid with

$$U_a = [1,0,0,0], \quad \mu = 3\mathcal{P} = \frac{3}{4t^2}, \tag{46}$$

which has a radiation equation of state. Transferring the $d = 4$ solution to spherical coordinates

$$ds^2 = -dt^2 + t\ d\Sigma_3^2, \quad \Sigma_3^2 = dr^2 + r^2(d\theta^2 + \sin(\theta)^2 d\phi^2), \tag{47}$$

grtensor calculates the Ricciscalar, Weyl, Bach and Pauli tensors all to vanish and the non-vanishing curvature is most easily expressed in terms of the Ricci scalar Φ_{00}

$$\Phi_{00} = 4\Phi_{11} = 4\Phi_{22} = \frac{1}{2t^2}, \tag{48}$$
$$RiemSq = 6\Phi_{00}, \quad RicciSq = 3\Phi_{00},$$
$$R_1 = \frac{3}{4}\Phi_{00}^2, \quad R_2 = \frac{3}{8}\Phi_{00}^3, \quad R_3 = \frac{21}{64}\Phi_{00}^4.$$

10 The quadratic order Kasner four brane.

For the line element 44 using the traditional constraints 16 with $a, b = 1$, the curvature products are

$$R = -\tfrac{5}{\alpha}, \quad RicciSq = \tfrac{1}{5}R^2, \tag{49}$$
$$WeylSq = \tfrac{16p_3^2(1-p_3)}{t^4}\exp(2\tfrac{\chi}{\sqrt{\alpha}}), \quad RiemSq = WeylSq + \tfrac{1}{10}R^2.$$

So far no solutions involving the Bach tensor have been found. The field equations

$$G_{ab} - 3P_{ab}/5 = 0, \tag{50}$$

and

$$G_{ab} + \alpha GB_{ab} + \alpha\delta_{ab}^{\chi\chi}WeylSq = 0, \tag{51}$$

are obeyed. For the expansion-free constraints $b = 0$ & $2a^2 = 3$ and the shear-free constraints $a^2 = b = 3$ the Bach tensor is non-vanishing, but no field equations have been found to be obeyed.

11 The double Kasner solution.

Instead of choosing a five space of constant curvature it is possible to choose the five space to be of Kasner form. An example of this is the double Kasner metric

$$ds_5^2 = \chi^{2q_1}t^{2p_1}dx^2 + \chi^{2q_2}t^{2p_2}dy^2 + \chi^{2q_3}t^{2p_3}dz^2 - \chi^{2q_4}t^{2p_4}dt^2 + \chi^{2q_5}t^{2p_5}d\chi^2, \tag{52}$$

which has Ricci tensor

$$\begin{aligned}
R_i{}^i &= \chi^{-2q_4}t^{-2p_4-2}p_i\left(P_5 - 2p_4 - 1\right) - \chi^{-2q_5-2}t^{-2p_5}q_i\left(Q_5 - 2q_5 - 1\right), \\
R_t{}^t &= \chi^{-2q_4}t^{-2p_4-2}\left(P_5^2 - p_4^2 + (p_4 - P_5)(1 + p_4)\right) - \chi^{-2q_5-2}t^{-2p_5}q_4\left(Q_5 - 2q_5 - 1\right), \\
R_\chi{}^\chi &= \chi^{-2q_5-2}t^{-2p_5}\left(Q_4(1 + q_5) - Q_4^2\right) - \chi^{-2q_4}t^{-2p_4-2}\left(-P_5 + 2p_4 + 1\right), \\
R_\chi{}^t &= \chi^{-2q_4-1}t^{-2p_4-1}\left(PQ - q_4P_3 - p_5Q_3\right),
\end{aligned} \tag{53}$$

where

$$PQ \equiv \sum_{i=1}^{3} q_ip^i, \quad P_j \equiv \sum_{i=1}^{j} p_i, \quad P_j^2 \equiv \sum_{i=1}^{j} p_i^2, \tag{54}$$

and similarly for q. There is the non-interacting vacuum solution

$$q_1 = q_2 = p_2 = p_3 = \frac{2}{3}, \quad p_1 = q_3 = -\frac{1}{3}, \quad p_4 = p_5 = q_4 = q_5 = 0. \tag{55}$$

There is no immediate interacting vacuum solution to these constraints, as for example

$$p_1 = p_2 = \frac{2}{3},\ p_3 = -\frac{1}{3},\ q_1 = q_2 = q_3 = \frac{1}{2},\ q_4 = -\frac{1}{2},\ p_4 = p_5 = q_5 = 0, \tag{56}$$

solves the non-interacting equations but gives the wrong sign for the interacting PQ equation. The choice

$$p_1 = p_2 = q_1 = q_2 = \frac{2}{3},\ p_3 = q_3 - \frac{1}{3},\ p_4 = -q_4 = -a,\ p_5 = -q_5 = 1-a, \tag{57}$$

where a is a constant, is a vacuum solution with non-vanishing Riemann components; however the Kretschmann invariant vanishes $RiemSq = K = 0$, also the quadratic tensors vanish.

12 Matter trapping

In order to consider matter trapping [11, 12] one takes test objects, such as particles, fields or fluids, and investigates whether they coalesce toward the brane or disperse. A criteria to judge whether this is happening whether the energy conditions [5]§4.3 or equations similar to them are obeyed. For example a minimal scalar field obeys the null converges condition, but this is not sufficient for minimal scalar fields always to coalesce toward the brane because the energy condition for the brane spacetime depends on its Ricci tensor rather the test minimal scalar's hatted Ricci tensor. Consider a massless variable separable Klein-Gordon test scalar field in the spacetime with line element given by 44 and 15, a solution is

$$\psi = At^{(1-b)}\mathrm{erf}\left(\alpha^{-\frac{1}{4}}\chi\right), \tag{58}$$

where A is a constant and b given by 16. For the null vector

$$N^a = \exp\left(-\frac{\chi}{\sqrt{\alpha}}\right)\delta^a_t \pm \delta^a_\chi \tag{59}$$

the null convergence condition is

$$\begin{aligned} N^a N^b \hat{R}_{ab} &= 2\left\{\exp\left(-\frac{\chi}{\sqrt{\alpha}}\right)\psi_t \pm \psi_\chi\right\}^2 \\ &= \frac{8A^2 t^{-2b}}{\pi\sqrt{\alpha}}\left\{\pm t + (1-b)\chi + \mathcal{O}\left(\chi^2\right)\right\}^2, \end{aligned} \tag{60}$$

as this is positive this particular Klein-Gordon matter converges toward the brane. It is not immediate what sort of energy conditions and vector fields are best in the non-null case, in particular should the vector field V be timelike or in some sense point toward the brane, and the usual energy conditions hold for an arbitrary timelike vector field, should this arbitrariness still be present or could a specific vector field be used. As a second test object consider a perfect fluid 6 with timelike vector field, acceleration and expansion

$$V_a = \exp\left(\frac{\chi}{2\sqrt{\chi}}\right)\delta^t_a, \quad \dot{V}^a = \frac{1}{2\sqrt{\alpha}}\delta^a_\chi, \quad \overset{V}{\theta} = -\frac{b}{t}\exp\left(-\frac{\chi}{2\sqrt{\alpha}}\right), \tag{61}$$

where b is given by 16. The conservation equations are

$$T_{t\cdot;b}^{\ \ b} = -\mu_t - \frac{b}{t}(\mu + \mathcal{P}), \quad T_{\chi\cdot;b}^{\ \ b} = \mathcal{P}_\chi + \frac{1}{2\sqrt{\alpha}}(\mu + \mathcal{P}), \tag{62}$$

assuming the equation of state 7 and separation of variables gives the solution

$$\mu = At^{-b\gamma}\exp\left(-\frac{\gamma\chi}{2(\gamma - 1)\sqrt{\alpha}}\right), \tag{63}$$

where A is a constant: this solution decays for large χ.

13 Conclusion.

For higher order tensors. The $d = 2$ Kasner spacetime 10 has a vacuum-Pauli solution when $p = -3$. No solutions to $d = 3$ Kasner spacetime have been found. The $d = 4$ Kasner spacetime is a vacuum-Einstein solution when the p's obey the traditional constraints, however there are additional vacuum-Bach solutions when the p's obey expansion-free or shear-free constraints 25. The $d = 5$ Kasner spacetime has a solution similar to the traditional $d = 4$ solution 29, and also a simple solution which has vanishing Gauss-Bonnet tensor 31. The Kasner one brane 33 is a solution to the vacuum-Einstein equations. The Kasner two brane is a solution to the Pauli-Einstein equations when $p = 1$. The Kasner three brane is a solution when $p_1 = p,\ p_2 = 0$, for $p = +1$ it is a solution to both the vacuum-Pauli and vacuum-Bach equations, for $p = -3$ it is a solution to the vacuum-Bach equations. The Kasner four brane has a solutions to the Pauli-Einstein 50 and also to the Gauss-Bonnet-Einstein equations 51.

For double Kasner spacetime 52 a non-interacting vacuum solution was found 55, but no interacting solution was found.

For brane fluids, with a perfect fluid in the brane and a vacuum in the bulk three brane 42 and four brane 46 solutions were found. The four brane solution has a four metric the same as $k = 0$ radiation filled Robertson-Walker spacetime.

For brane fluids, a test scalar field obeys a null convergence condition 61, and in this sense the field converges onto the brane; a test perfect fluid 63 decays away from the brane and in this sense is trapped on the brane.

References

[1] R. Bach *Math.Zeit.*9(1921),110. 2

[2] John D. Barrow & Christos G. Tsagas
The Gödel Brane, `gr-qc/0309030` *Phys.Rev.D* **69**(2004)064007. 2

[3] Alessandro Fabbri, David Langlois, Daniele A. Steer & Robin Zegers
Brane cosmology with an anisotropic bulk, `hep-th/0407262`.

[4] Andrei V. Frolov
Kasner-AdS spacetime and anisotropic brane-world cosomology, `gr-qc/0102064`.

[5] S.W.Hawking and G.F.R.Ellis
The large scale structure of space-time. CUP (1973) ISBN 0 521 09906 4 9

[6] R.J. van den Hoogen, A.A. Coley, Y. He
Bianchi type IX brane-world cosmologies. *Phys.Rev.D* **68**(2003)023502. 2

[7] Peter Musgrave, Dennis Pollney & Kayll Lake
(1996) GRTensorII, http://astro.queens.ca/∼grtensor/ 2

[8] W. Pauli *Physik.Zeit.*(1919),**20**,457,eq.50. 2

[9] L. Randall & R. Sundrum
*Phys.Rev.Lett.***83**(1999)3370,4690. `hep-th/9906064` 2

[10] Mark D. Roberts
Interacting with the fifth dimension. `gr-qc/0412051` 5

[11] V.A. Rubakov and M.E. Shaposhnikov
Do we live inside a domain wall? *Phys.Lett.B***125**(1983)136. 9

[12] Tracy R. Slatyer and Raymond R. Volkas
Cosmology and fermion confinement in a scalar-field-generated domain wall brane in five dimensions. *JHEP* **04**(2007)062. 9

[13] Barton Zwiebach
Curvature squared terms and string theories. *Phys.Lett.B* **156**(1985)1315. 2