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A RELATIVISTIC MODEL FOR STRANGE QUARK STARS

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We propose a spherically symmetric and anisotropic model for strange quark stars within the framework of MIT Bag model. Though the model is found to comply with all the physical requirements of a realistic star satisfying a strange matter equation of state (EOS), the estimated values the Bag constant for different strange star candidates like Her X-1, SAX J 1808.4-3658 and 4U 1820-30, clearly indicate that the Bag constant need not necessarily lie within the range of $60 - 80 \text{ MeV fm}^{-3}$ as claimed in the literature^{1,2}.

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1. Introduction

In 1964, Gell-Mann³ and Zweig⁴ had independently suggested that hadrons are composed of even more fundamental particles called quarks, a proposition which got experimental support later on. In the theoretical front, the quark matter hypothesis put forward by Witten⁵, had prompted many investigators to investigate an entirely new class of compact astrophysical objects composed of strange quark matter called strange stars (see Weber⁶ for a recent review). As quarks are not seen as free particles, the quark confinement mechanism have been dealt with great details in QCD. In the MIT bag model⁷ for strange stars, the quark confinement has been assumed to be caused by a universal pressure B , called the bag constant. Farhi and Jafee¹ and Alcock *et al*² had shown that for a stable strange quark matter the value of the bag constant should be $B \sim 55 - 75 \text{ MeV fm}^{-3}$.

At the backdrop of such theoretical developments, analytic model building of strange quark stars naturally plays a crucial role to understand the gravitational behaviour of strange stars. In this vein, from the perspective of classical GR, we propose here an analytic solution capable of describing realistic strange stars. In an earlier work, some of us⁸ had proposed a relativistic model for strange stars where the geometric model of Krori and Barua⁹ was used to describe the interior space-time of strange stars. In this paper, we adopt the Finch and Skea¹³ model to develop the interior space-time of a strange star. We assume a spherically symmetric star composed of strange quark matter whose equation of state (EOS) is governed by the MIT bag model. We also assume anisotropy in the matter composition implying that the radial pressure (p_r) is not equal to the tangential pressure (p_t). Since the density of a strange star may exceed the nuclear matter density it is expected that pressure at the interior of such would be anisotropic, in general^{10,11}. Anisotropy in a highly dense compact stellar object may occur for various reasons (e.g, existence of a solid core, phase transition, presence of electromagnetic field etc.) whose contributions in the stellar modelling can be incorporated by assuming the star to be anisotropic in general¹². In our construction, the assumption of anisotropy provides an extra degree of freedom to deal with the system.

The strange star model, in this work, has been developed by considering the MIT bag model EOS and a particular ansatz for the metric function g_{rr} proposed by Finch and Skea¹³. The solution found here has been shown to comply with all the physical requirements of a realistic star. When applied to some proposed strange star candidates, it has been found that the corresponding values of the bag constant lie on the higher side as compared to its acceptable range put forward by Farhi and Jafee¹ and Alcock *et al*². In fact, our results are consistent with the experimental results from CERN-SPS and RHIC, indicating a wider range of values of the bag constant¹⁴.

2. Interior solution:

We assume that the interior space-time of a strange star is described by a spherically symmetric metric of the form

$$ds^2 = -e^{\nu(r)} dt^2 + \left(1 + \frac{r^2}{R^2}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where, the metric function $\nu(r)$ is yet to be determined. Note that the ansatz for the metric function g_{rr} in (1) was proposed by Finch and Skea¹³ to develop a viable model for a relativistic compact star. The $t = \text{constant}$ hyper-surface of the metric is paraboloidal in nature which exhibits a departure from spherical geometry and the constant R is a curvature parameter which governs the geometry of the background space-time.

We assume that the energy-momentum tensor for the strange matter filling the interior of the star has the standard form $T_{ij} = \text{diag}(\rho, -p_r, -p_t, -p_t)$, where, ρ is the energy-density; p_r and p_t are the radial and transverse pressures, respectively. Einstein's field equations for the line-element (1), accordingly, are obtained as (we set $G = c = 1$)

$$8\pi\rho = \frac{1}{R^2} \left(3 + \frac{r^2}{R^2}\right) \left(1 + \frac{r^2}{R^2}\right)^{-2}, \quad (2)$$

$$8\pi p_r = \left(1 + \frac{r^2}{R^2}\right)^{-1} \left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2}, \quad (3)$$

$$8\pi p_t = \left(1 + \frac{r^2}{R^2}\right)^{-1} \left[\frac{\nu''}{2} + \frac{\nu'}{2r} + \frac{\nu'^2}{4}\right] - \frac{1}{R^2} \left(1 + \frac{r^2}{R^2}\right)^{-2} \left[1 + \frac{\nu' r}{2}\right]. \quad (4)$$

Eqs. (2)-(4) constitute a system of four unknowns (ρ, p_r, p_t, ν). By suitably choosing any one of these unknown parameters, the system may be solved. Our objective here is to develop a model for strange stars and, therefore, we assume the simplest form of the strange matter EOS having the form

$$p_r = \frac{1}{3}(\rho - 4B), \quad (5)$$

where, B is the bag constant. Substituting Eq. (5) in Eq. (3) and integrating, we determine the unknown metric function ν in the form

$$\nu = \frac{1}{3} \ln(R^2 + r^2) - \frac{8\pi B}{3} \frac{r^2}{R^2} \left(2R^2 + r^2 - \frac{1}{4\pi B}\right) + \nu_0. \quad (6)$$

where ν_0 is an integration constant.

Using Eq. (6) in Eqs. (2)-(4), the density and the two pressures are then obtained

4 *Kalam et al*

as

$$8\pi\rho = \frac{1}{R^2} \frac{\left(3 + \frac{r^2}{R^2}\right)}{\left(1 + \frac{r^2}{R^2}\right)^2}, \quad (7)$$

$$8\pi p_r = \frac{1}{3R^2} \frac{\left(3 + \frac{r^2}{R^2}\right)}{\left(1 + \frac{r^2}{R^2}\right)^2} - \frac{32\pi B}{3}, \quad (8)$$

$$\begin{aligned} 8\pi p_t = & \left(1 + \frac{r^2}{R^2}\right)^{-1} \left[\frac{256\pi^2 B^2 R^2}{9} \left(\frac{r}{R}\right)^6 + \left(\frac{512\pi^2 B^2 R^2}{9} - \frac{64\pi B}{9} - \frac{32\pi B}{9\left(1 + \frac{r^2}{R^2}\right)} \right) \left(\frac{r}{R}\right)^4 \right. \\ & + \left(\frac{256\pi^2 B^2 R^2}{9} - \frac{64\pi B}{9} + \frac{4}{9R^2} - \frac{2}{9R^2\left(1 + \frac{r^2}{R^2}\right)^2} + \frac{1}{R^2\left(1 + \frac{r^2}{R^2}\right)} - \frac{32\pi B}{9\left(1 + \frac{r^2}{R^2}\right)} + \frac{4}{3r^2} \right) \left(\frac{r}{R}\right)^2 \\ & + \frac{1}{3R^2\left(1 + \frac{r^2}{R^2}\right)^2} + \frac{1}{3R^2\left(1 + \frac{r^2}{R^2}\right)} - \frac{32\pi B}{3} \left. \right] - \left(1 + \frac{r^2}{R^2}\right)^{-2} \left[\frac{2}{3R^2} \left(\frac{3}{2} + \frac{r^2}{R^2}\right) \right. \\ & \left. + \frac{1}{3R^2} \frac{\left(\frac{r}{R}\right)^2}{\left(1 + \frac{r^2}{R^2}\right)} - \frac{16\pi B}{3} \left(\frac{r}{R}\right)^2 \left(1 + \frac{r^2}{R^2}\right) \right]. \quad (9) \end{aligned}$$

3. Physical behaviour of the model

We note that the physical behaviour of the model depends on the constants R and B . We need to put appropriate bounds on these parameters so that the model can describe a realistic strange star. To this end, based on various physical requirements, let us now analyze the behaviour of the physical parameters.

Let us assume that b be the radius of the star. Then, from Eq. (8), the central and surface densities are respectively obtained as

$$\rho_0 = \frac{3}{8\pi R^2}, \quad (10)$$

$$\rho_b = \frac{1}{8\pi R^2} \left(3 + \frac{b^2}{R^2}\right) \left(1 + \frac{b^2}{R^2}\right)^{-2}. \quad (11)$$

We also have

$$\frac{d\rho}{dr} = -\frac{r\left(5 + \frac{r^2}{R^2}\right)}{4\pi R^4\left(1 + \frac{r^2}{R^2}\right)^3} < 0, \quad (12)$$

$$\frac{d\rho}{dr}(r=0) = 0, \quad (13)$$

$$\frac{d^2\rho}{dr^2}(r=0) = -\frac{5}{4\pi R^4} < 0. \quad (14)$$

Obviously, the density is maximum at the centre and it decreases radially outward.

Similarly, from Eq. (8), we have

$$\frac{dp_r}{dr} = -\frac{r(5 + \frac{r^2}{R^2})}{12\pi R^4(1 + \frac{r^2}{R^2})^3} < 0, \quad (15)$$

$$\frac{dp_r}{dr}(r=0) = 0, \quad (16)$$

$$\frac{d^2p_r}{dr^2} = -\frac{5}{12\pi R^4} < 0, \quad (17)$$

which show that the radial pressure also decreases from the centre towards the boundary. Thus, the energy density and the radial pressure are well behaved in the interior of the stellar configuration. Variations of the energy-density and two pressures have been shown in Fig. (1) and (2), respectively.

The anisotropic parameter $\Delta(r) = \frac{2}{r}(p_t - p_r)$ is obtained as

$$\begin{aligned} \Delta = & \frac{1}{4\pi r} \left(1 + \frac{r^2}{R^2}\right)^{-1} \left[\frac{256\pi^2 B^2 R^2}{9} \left(\frac{r}{R}\right)^6 + \left(\frac{512\pi^2 B^2 R^2}{9} - \frac{64\pi B}{9} - \frac{32\pi B}{9(1 + \frac{r^2}{R^2})} \right) \left(\frac{r}{R}\right)^4 \right. \\ & + \left(\frac{256\pi^2 B^2 R^2}{9} - \frac{64\pi B}{9} + \frac{4}{9R^2} - \frac{2}{9R^2(1 + \frac{r^2}{R^2})^2} + \frac{1}{R^2(1 + \frac{r^2}{R^2})} - \frac{32\pi B}{9(1 + \frac{r^2}{R^2})} + \frac{4}{3r^2} \right) \left(\frac{r}{R}\right)^2 \\ & \left. + \frac{1}{3R^2(1 + \frac{r^2}{R^2})^2} + \frac{1}{3R^2(1 + \frac{r^2}{R^2})} - \frac{32\pi B}{3} \right] \\ & - \frac{1}{4\pi r} \left(1 + \frac{r^2}{R^2}\right)^{-2} \left[\frac{2}{3R^2} \left(\frac{3}{2} + \frac{r^2}{R^2}\right) + \frac{1}{3R^2} \frac{(\frac{r}{R})^2}{(1 + \frac{r^2}{R^2})} - \frac{16\pi B}{3} \left(\frac{r}{R}\right)^2 \left(1 + \frac{r^2}{R^2}\right) \right. \\ & \left. - \frac{1}{12\pi r R^2} \frac{(3 + \frac{r^2}{R^2})}{(1 + \frac{r^2}{R^2})^2} + \frac{8B}{3r} \right] \end{aligned} \quad (18)$$

Figure (3) shows the nature of the anisotropic stress at the stellar interior for a particular case. The condition that the anisotropic parameter Δ should vanish at the centre ($r=0$) yields

$$\frac{1}{8\pi R^2} - \frac{4}{3}B = 0, \quad (19)$$

which can be used to calculate the Bag constant B .

3.1. Matching conditions

At the boundary of the star $r = b$, the interior metric should be matched to the Schwarzschild exterior metric. Continuity of the metric functions across the bound-

6 *Kalam et al*

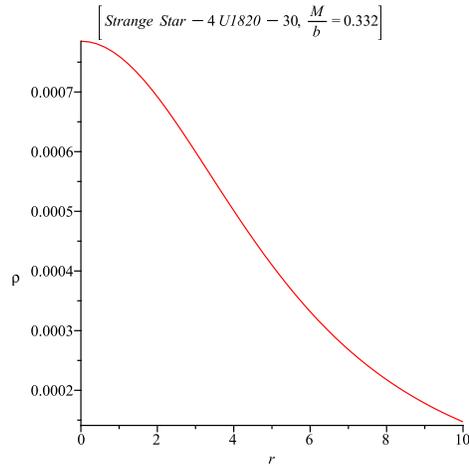


Fig. 1. Variation of the energy-density (ρ) at the interior of the star.

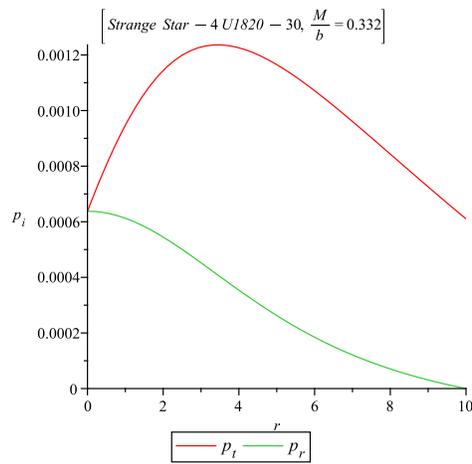


Fig. 2. Variation of the radial(p_r) and transverse pressure(p_t) at the interior of the star.

ary surface yields

$$\left(1 + \frac{b^2}{R^2}\right)^{-1} = 1 - \frac{2M}{b}, \quad (20)$$

$$\begin{aligned} \nu(r=b) &= \ln\left(1 - \frac{2M}{b}\right) = \frac{1}{3} \ln(R^2 + b^2) \\ &\quad - \frac{8\pi B}{3} \frac{b^2}{R^2} \left(2R^2 + b^2 - \frac{1}{4\pi B}\right). \end{aligned} \quad (21)$$

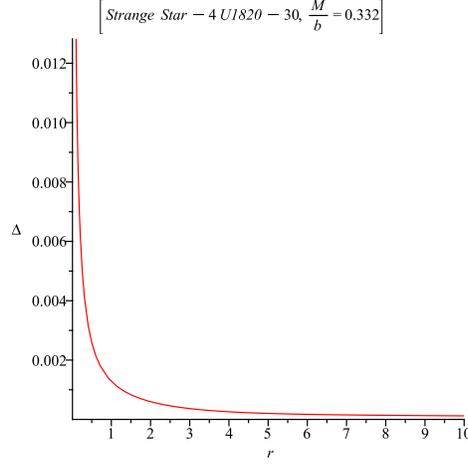


Fig. 3. Variation of the anisotropic stress $\Delta(r) = \frac{2}{r} (p_t - p_r)$ at the stellar interior.

From Eqs. (20), we obtain the compactness of the star as

$$\frac{M}{b} = \frac{b^2}{2R^2} \left(1 + \frac{b^2}{R^2}\right)^{-1}. \quad (22)$$

The condition that the radial pressure must vanish at the boundary ($p_r(r = b) = 0$) determines the bag constant in the form

$$B = \frac{1}{32\pi R^2} \frac{\left(3 + \frac{b^2}{R^2}\right)}{\left(1 + \frac{b^2}{R^2}\right)^2}. \quad (23)$$

3.2. TOV equation

For an anisotropic fluid distribution, the generalized Tolman-Oppenheimer-Volkoff (TOV) equation gets the form

$$\frac{dp_r}{dr} + \frac{1}{2}\nu'(\rho + p_r) + \frac{2}{r}(p_r - p_t) = 0. \quad (24)$$

We rewrite the above equation in the form

$$-\frac{M_G(\rho + p_r)}{r^2} e^{\frac{\lambda-\nu}{2}} - \frac{dp_r}{dr} + \frac{2}{r}(p_t - p_r) = 0, \quad (25)$$

where, $M_G(r)$ is the effective gravitational mass inside a sphere of radius r and is given by

$$M_G(r) = \frac{1}{2}r^2 e^{\frac{\nu-\lambda}{2}} \nu'. \quad (26)$$

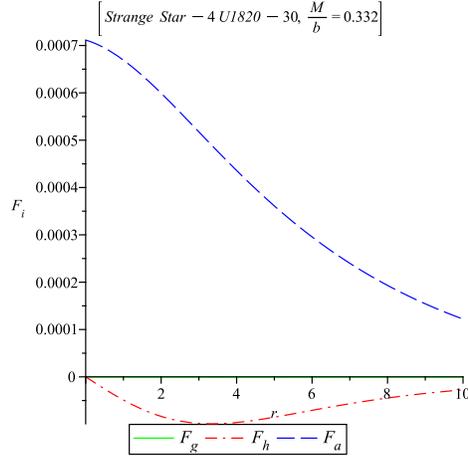


Fig. 4. Three different forces acting on the fluid elements of a given configuration.

The modified TOV equation describes the equilibrium condition for the strange star subject to gravitational (F_g), hydrostatic (F_h) and anisotropic (F_a) stresses within the stellar interior so that

$$F_g + F_h + F_a = 0, \quad (27)$$

where the stress components are given by

$$F_g = -Br(\rho + p_r), \quad (28)$$

$$F_h = -\frac{dp_r}{dr}, \quad (29)$$

$$F_a = \frac{2}{r}(p_t - p_r). \quad (30)$$

In Fig. 4, we have shown variations of F_g , F_h and F_a for a particular stellar configuration. Obviously, it is possible to construct a static equilibrium configuration in the presence of anisotropic, gravitational and hydrostatic stresses.

3.3. Energy conditions

Imposition of the energy conditions, namely, the null (NEC), weak (WEC), strong (SEC) and dominant (DEC) energy conditions, put certain bounds on the model parameters. In our model, applying these energy conditions at the centre ($r = 0$), we get the following bounds:

- (i) NEC: $p_0 + \rho_0 \geq 0 \Rightarrow B \leq \frac{3}{8\pi R^2}$, i.e. $B \leq \rho_0$,
- (ii) WEC: $p_0 + \rho_0 \geq 0 \Rightarrow B \leq \frac{3}{8\pi R^2}$, i.e. $B \leq \rho_0$, $\rho_0 \geq 0$,
- (iii) SEC: $p_0 + \rho_0 \geq 0 \Rightarrow B \leq \rho_0$, $3p_0 + \rho_0 \geq 0 \Rightarrow B \leq \frac{\rho_0}{2}$,
- (iv) DEC: $\rho_0 > |p_0| \Rightarrow B \leq \frac{\rho_0}{2}$.

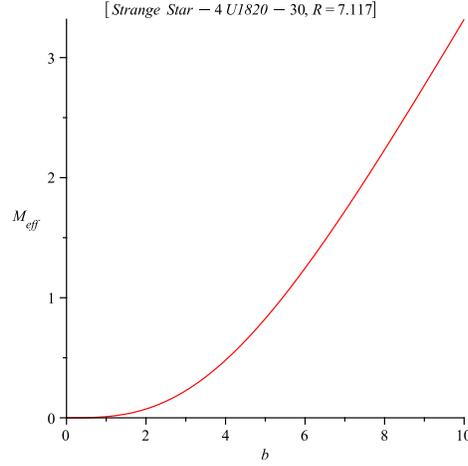


Fig. 5. Variation of M_{eff} against radial parameter r .

Values of the model parameters for different stellar configurations are in agreement with these bounds as shown in Table 2.

3.4. Mass-Radius relation

For a static spherically symmetric perfect fluid star in equilibrium, Buchdahl¹⁵ showed that the maximum allowed mass-radius ratio is given by $\frac{2M}{R} < \frac{8}{9}$ (for a more generalized expression see Mak *et al*¹⁶). In our model, the effective gravitational mass is obtained as

$$M_{eff} = 4\pi \int_0^b \rho r^2 dr = \frac{b}{2} \left[\frac{\frac{b^2}{R^2}}{1 + \frac{b^2}{R^2}} \right]. \quad (31)$$

In Fig. 5, the effective mass for a given radius has been shown. The compactness of the star, accordingly, is given by

$$u = \frac{M_{eff}(b)}{b} = \frac{1}{2} \left[\frac{\frac{b^2}{R^2}}{1 + \frac{b^2}{R^2}} \right], \quad (32)$$

whose values at different radii have been shown in Fig. 6. We note that the constraint on the maximum allowed mass-radius ratio in our case is similar to the isotropic fluid sphere, i.e., $\frac{M}{b} < \frac{4}{9}$. The corresponding surface redshift (Z_s) is obtained as

$$Z_s = [1 - (2u)]^{-\frac{1}{2}} - 1 = \sqrt{1 + \frac{b^2}{R^2}} - 1. \quad (33)$$

The maximum surface redshift for different strange star candidates in this model has been shown in Table 1.

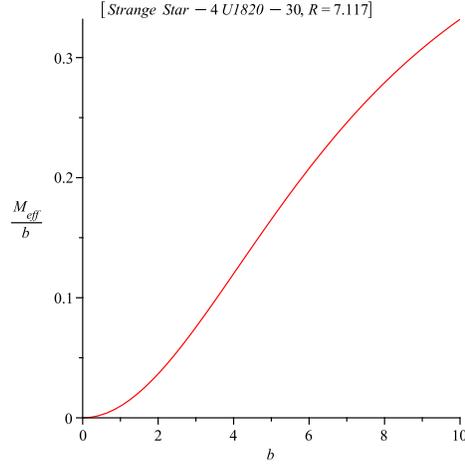


Fig. 6. Variation of $\frac{M_{eff}}{b}$ against radial parameter r .

Table 1. Values of the maximum surface redshift for different strange star candidates.

Strange star candidate	b (km)	R (km)	$Z_s(\text{max})$
Her X-1	7.7	10.79	0.2285
SAX J 1808.4-3658(SS1)	7.07	5.787	0.5787
SAX J 1808.4-3658(SS2)	6.35	5.0282	0.6108
4U 1820-30	10.0	7.117	0.7246

3.5. Estimated bag values of some strange star candidates

Based on the analytic model developed so far, to get an estimate of the range of various physical parameters, let us now consider some strange star candidates like 4U 1820-30, Her X-1 and SAX J 1808.4-3658. Assuming the estimated mass and radius of these stars, we have calculated the values of the relevant physical parameters which have been compiled in Table 2. In Table 3, we have expressed the values of the bag constant, in particular, for respective stellar configurations. We observe that a wide range of values of the bag constant are possible unless we impose some (yet unknown) constraints on B . However, from the perspective of a mathematically self-consistent model, it appears that a wide range of values of the bag constant are possible which is consistent with the CERN-SPS and RHIC data.

4. Conclusion

We have obtained a new class of solutions for the interior of a compact stellar object like a strange star. To construct the model, we have used the phenomenological MIT bag model EOS for quark matter. The analytic solution obtained is non-singular

Table 2. Values of the model parameters for different Strange stars. (Data obtained for the star 4U 1820-30 have been utilized to plot figures.)

Strange star candidate	$M (M_{\odot})$	b (km)	$\frac{M}{b}$	R (km)	B (km $^{-2}$)	ρ_0 (km $^{-2}$)	ρ_b (km $^{-2}$)	p_0 (km $^{-2}$)
Her X-1	0.88	7.7	0.168	10.79	0.00013	0.00102	0.00052	0.000165
SAX J 1808.4-3658(SS1)	1.435	7.07	0.299	5.787	0.00021	0.0035	0.00085	0.000901
SAX J 1808.4-3658(SS2)	1.323	6.35	0.308	5.0282	0.00026	0.00472	0.00107	0.00121
4U 1820-30	2.25	10.0	0.332	7.117	0.0001	0.0023	0.00044	0.00063

Table 3. Physical values of energy density, pressure and Bag constant for different Strange stars.

Strange star candidate	Central density (gm cm $^{-3}$)	Surface density (gm cm $^{-3}$)	Central pressure (dyne cm $^{-2}$)	Bag constant (MeV fm $^{-3}$)
Her X-1	1.38169×10^{15}	0.710012×10^{15}	$4.145070271 \times 10^{35}$	99.7207866
SAX J 1808.4-3658(SS1)	$4.808527413 \times 10^{15}$	1.159209×10^{15}	$14.42558225 \times 10^{35}$	162.8102741
SAX J 1808.4-3658(SS2)	$6.370476078 \times 10^{15}$	$1.449092801 \times 10^{15}$	$19.11142825 \times 10^{35}$	203.5242699
4U 1820-30	$3.179364432 \times 10^{15}$	$0.5960016695 \times 10^{15}$	$9.538093302 \times 10^{35}$	83.70809967

and is anisotropic in nature. The estimated values of the bag constant B for different strange star candidates have been found to be on the higher side as compared to its estimated range of 60 – 80 MeV fm $^{-3}$ for a β -equilibrium stable strange matter configuration^{1,2}. It is likely that a wide range of values of the bag constant are permissible which is in agreement with the recent CERN-SPS and RHIC data.

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12 *Kalam et al*

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