

All order covariant tubular expansion

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Abstract

We consider tubular neighborhood of an arbitrary submanifold embedded in a (pseudo-)Riemannian manifold. This can be described by Fermi normal coordinates (FNC) satisfying certain conditions as described by Florides and Synge in [15]. By generalizing the work of Muller *et al* in [54] on Riemann normal coordinate expansion, we derive all order FNC expansion of vielbein in this neighborhood with closed form expressions for the curvature expansion coefficients. Our result is shown to be consistent with certain integral theorem for the metric proved in [15].

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1 Motivation and summary

As it is well known, Riemann normal coordinate (RNC) system is a very useful tool in differential geometry and general relativity [1]-[4]. In particular, it is used for computing covariant Taylor expansion of tensors around a point in a manifold [5]-[10].

Fermi normal coordinate (FNC) system is a generalization of RNC in the sense that the RNC-origin, which may be viewed as a zero-dimensional submanifold of the ambient space, is replaced by a higher dimensional one¹. It is related to a general coordinate system through the exponential map along the normal directions. The region surrounding the submanifold where FNC is well defined, i.e. where the exponential map is a diffeomorphism, is called a *tubular neighborhood*. The existence theorem for such a neighborhood suffices as a powerful tool for various analysis in differential geometry [16, 17].

The use of FNC can be found in various areas of physics. Some of them, which we will briefly discuss below, are general relativity, multi-particle and string dynamics in curved spacetime and *constrained quantum mechanics*. In all such applications one needs to know

¹FNC was first introduced by E. Fermi for a curve in [11]. It was then followed by various other generalizations (see [12]-[14] and references therein). Our consideration, which is same as that of [15], will be described in detail in §2.

explicitly the covariant Taylor expansion coefficients of geometric quantities around the submanifold. The same is true for certain applications outside physics also. For example, see [18] which describes how this is used in computing volume of tubes [19]-[21].

In general relativity one is mainly interested in studying various physical effects around a particle worldline (one dimensional submanifold) in a given background [22]-[42]. In one line of research ([32]-[41]) FNC expansion around a worldline has been used to compute curvature corrections to the energy spectrum of hydrogen-like atoms in curved backgrounds. The reason for considering a worldline is the following approximation (see also [42]): the center of mass (CM) of the system falls freely along a geodesic during the time interval relevant for the study of the system (e.g. an atomic transition).

The above assumption is physically well-motivated and should work fine when the CM-mass is large. However, it is also a simplifying assumption. Going beyond this assumption requires a drastic change in the analysis where a different tubular geometry around a higher dimensional submanifold becomes relevant. Let us explain this slightly more elaborately. Physically one should expect that the internal dynamics of the atom (or any other composite object) backreacts to the motion of the CM. In other words, it should be possible to derive the geodesic equation for CM at the leading order in a semi-classical expansion² where the CM mass is large. Notice that this is a relativistic bound state problem in curved spacetime. A completely satisfactory description of such a problem has not yet been understood even in flat spacetime [43, 44]. However, one may expect that the basic geometric structure underlying any manifestly covariant description of an n -particle bound state problem in M is the tubular geometry around $\Delta \hookrightarrow M^n$. Here $M^n := M \times M \times \dots$ (n -times) is the n -particle (extended) configuration space and $\Delta (\cong M)$ is the diagonal submanifold of all possible locations of the CM (assuming equal mass for all the particles). The reason why Δ is the subspace where the CM lies has been explained in [45].

Recently the above approach has been pursued in an analogous system in [45]. Here the analogue of a multi-particle bound configuration is a closed string in M , the configuration space is the corresponding loop space LM , the relativistic quantum theory is the loop space quantum mechanics [45]-[48] and the analogue of (CM-mass)⁻¹ is the parameter α' . It was shown that in a semi-classical limit ($\alpha' \rightarrow 0$) the string wavefunction localizes

²What we have in mind is an expansion in inverse of the CM-mass. The quantum mechanical \hbar may be set to 1.

on $\Delta(\cong M)$ - the subspace of string CM, which is same as the space of vanishing loops. Moreover, the semi-classical expansion is obtained from the tubular expansion of geometric quantities around $\Delta \hookrightarrow LM$.³

Another application of tubular geometry around a higher dimensional submanifold is found in the context of what has been termed in the literature as constrained quantum systems [49, 50]. Here one considers a non-relativistic classical system in an ambient space with a potential that tries to confine the motion into a submanifold. The idea is to realize this constraint at the quantum mechanical level through localization of wavefunction [51]-[53]. The mechanism of finding the effective theory on the submanifold is very similar to that of the string case discussed above.

As mentioned earlier, the basic raw data that goes into all the above computations are the tubular expansion coefficients of geometric quantities around the submanifold. So far only a few such coefficients have been known. The goal of this paper is to obtain closed form expression for the expansion coefficients of vielbein to all orders in a completely generic situation where an arbitrary submanifold is embedded in a higher dimensional space.

We now briefly discuss the technical points relevant to our analysis. In [15], Florides and Synge (FS) constructed the special coordinate system under consideration for an arbitrary submanifold embedded in a higher dimensional (pseudo-)Riemannian space⁴. They wrote down the special coordinate conditions in terms of the metric and proved an integral theorem describing its behavior away from the submanifold. We will review the basic results in §2.

In [54], Muller, Schubert and van de Ven considered the RNC coordinate conditions written in terms of the vielbein and spin connection. Using certain differential geometric techniques it was possible to write down an integral equation for the vielbein in terms of the Riemann curvature tensor⁵. The authors were able to solve this equation to produce the complete RNC expansion of vielbein with a closed form covariant expression for the curvature expansion coefficients.

³Therefore, according to this analogy the corrections to the CM motion of any quantum mechanical bound state in M are analogous to α' corrections in string theory.

⁴The authors of [15] called it a *submanifold based coordinate system*. However, following the modern nomenclature (see, for example [18]) we will continue to call it FNC.

⁵As mentioned in [54], such coordinate conditions are the gravity analogue of the Fock-Schwinger gauge [55] in gauge theory which can also be used to write down an integral equation for the gauge potential in terms of the field strength [56].

Here we will use the same techniques to generalize the results of [54] to the case of FNC. In particular, we derive the integral equation for the vielbein and closed form covariant expressions for its curvature expansion coefficients in the set up considered in [15]. The results are different when the vector index of the vielbein takes values along the directions tangential and transverse to the submanifold. The transverse results are exactly the same as that of [54], as expected. This will be discussed in §3. In §4, we show how our result is consistent with the metric integral theorem of [15]. We discuss a demonstrative example in §5 and finally conclude in §6. Explicit numerical results for the expansion of vielbein and metric up to 10-th order in FNC have been presented in one of the appendices.

2 Metric-integral-theorem due to Florides and Synge

We consider an arbitrary D -dimensional submanifold M embedded in a higher dimensional pseudo-Riemannian space L of dimension d . Our notation for indices is as follows: Greek indices (α, β, \dots) run over D dimensions, capital Latin indices (A, B, \dots) run over $(d - D)$ transverse dimensions and small Latin indices (a, b, \dots) , over all dimensions. Other notations and conventions essential for the rest of our discussion are presented in appendix A.

The work of [15] proved the existence of a coordinate system $z^a = (x^\alpha, y^A)$ (that will be called FNC following [18]), where x^α is a general coordinate system in M whose embedding is given by,

$$y^A = 0 , \tag{2.1}$$

and the following conditions are satisfied for the transverse coordinates,

$$g_{aB}(x, y)y^B = \eta_{aB}y^B , \tag{2.2}$$

where $g_{ab}(x, y)$ is the metric tensor in FNC and η_{ab} is as define in eq.(A.63).

The following integral theorem was proved in [15],

$$\begin{aligned} g_{AB}(x, y) &= \eta_{AB} + 2y^C y^D \int_0^1 dt F_1(t) l_{ACDB}(x, ty) , \\ g_{A\beta}(x, y) &= y^C \underline{g_{A\beta, C}} + 2y^C y^D \int_0^1 dt F_2(t) l_{ACD\beta}(x, ty) , \end{aligned}$$

$$g_{\alpha\beta}(x, y) = G_{\alpha\beta}(x) + y^C \underline{g_{\alpha\beta,C}} + 2y^C y^D \int_0^1 dt F_3(t) l_{\alpha C D \beta}(x, ty) . \quad (2.3)$$

For any function $f(x, y)$ in L , we have defined: $\underline{f} \equiv f(x, 0)$. A comma in the suffix indicates ordinary derivative with respect to the argument. For example,

$$\underline{g_{\alpha\beta,C}} = \lim_{y \rightarrow 0} \frac{\partial}{\partial y^C} g_{\alpha\beta}(x, y) . \quad (2.4)$$

$G_{\alpha\beta} = \underline{g_{\alpha\beta}}$ is the induced metric on M and the functions $F_i(t)$ ($i = 1, 2, 3$) are defined as follows,

$$F_1(t) = t(1 - t) , \quad F_2(t) = \frac{1}{2}(1 - t^2) , \quad F_3(t) = 1 - t . \quad (2.5)$$

Finally,

$$l_{acdb}(x, y) = \frac{1}{2}(g_{ab,cd} + g_{cd,ab} - g_{ad,cb} - g_{cb,ad})(x, y) , \quad (2.6)$$

is the linear part of the covariant Riemann curvature tensor.

3 FNC expansion in arbitrary tubular neighbourhood

The above metric-integral-theorem involves the linear part of the curvature tensor. This form is not very useful for deriving the curvature expansion. Below we will use the technique of [54] to derive the curvature expansion for vielbein. In §4 we will show how our result is consistent with that of FS.

3.1 Integral equations for vielbein

We will first derive the integral equations satisfied by the vielbein (eqs.(3.23) below). To this end, we define, following [54], the *radial vector field* (see appendix A for our notations),

$$\mathbf{r} = y^A \mathcal{E}_A . \quad (3.7)$$

Then the coordinate condition (2.2) can be rewritten in terms of the vielbein and the connection one-form in the following way,

$$\mathbf{r} \hat{\mathcal{E}}^{(a)} = \delta_B^a y^B , \quad (3.8)$$

$$\mathbf{r} \omega^{(a)}_{(b)} = 0 , \quad (3.9)$$

where \mathbf{i}_r denotes the interior product [57] with respect to the vector field \mathbf{r} and δ^a_b is the Kronecker delta function⁶.

The next step is to relate the second order Lie derivative of $\hat{\mathcal{E}}^{(a)}$ and the curvature two-form. Using eqs. (A.66, 3.9) one writes,

$$\mathcal{L}_r \hat{\mathcal{E}}^{(a)} = \omega^{(a)}{}_{(b)} \mathbf{i}_r \hat{\mathcal{E}}^{(b)} + d\mathbf{i}_r \hat{\mathcal{E}}^{(a)} . \quad (3.12)$$

Then using the following result, which can be easily proved by using (3.8),

$$\mathcal{L}_r d\mathbf{i}_r \hat{\mathcal{E}}^{(a)} = d\mathbf{i}_r \hat{\mathcal{E}}^{(a)} , \quad (3.13)$$

one arrives at,

$$\mathcal{L}_r (\mathcal{L}_r - 1) \hat{\mathcal{E}}^{(a)} = \mathcal{L}_r \omega^{(a)}{}_{(b)} \mathbf{i}_r \hat{\mathcal{E}}^{(b)} . \quad (3.14)$$

Finally, one calculates two sides of the above equation independently. The left hand side is directly calculated by noting,

$$\hat{\mathcal{E}}^{(a)} = e^{(a)}{}_\alpha(x, y) \mathcal{E}^\alpha + e^{(a)}{}_A(x, y) \mathcal{E}^A , \quad (3.15)$$

and the right hand side can be calculated by using eqs.(A.67, 3.9). This leads to the following second order differential equation for the vielbein,

$$\mathbf{d}(\mathbf{d} + \epsilon^b) e^{(a)}{}_b(x, y) = \rho^{(a)}{}_{(c)}(x, y; y) e^{(c)}{}_b(x, y) , \quad (3.16)$$

where

$$\rho^{(a)}{}_{(b)}(x, y; \tilde{y}) \equiv r^{(a)}{}_{CD(b)}(x, y) \tilde{y}^C \tilde{y}^D . \quad (3.17)$$

We convert the non-coordinate to coordinate indices and *vice-versa* by using the vielbein and its inverse. For example,

$$r^{(a)}{}_{CD(b)} = r^{(a)}{}_{(c)(d)(b)} e^{(c)}{}_C e^{(d)}{}_D . \quad (3.18)$$

⁶To see how (3.8, 3.9) and (2.2) are equivalent one first shows that (2.2) implies,

$$\gamma^a{}_{BC}(x, y) y^B y^C = 0 , \quad (3.10)$$

and *vice versa* ($\gamma^a{}_{bc}$ being the Christoffel symbols as defined in eq.(A.64)). One then rewrites the above equations in terms of the vielbein and spin connection coefficients to derive (3.9) and,

$$\partial_A e^{(a)}{}_B(x, y) y^A y^B = 0 , \quad (3.11)$$

which, in turn, is solved by (3.8).

Moreover, for any function $f(x, y)$ we have defined,

$$\mathbf{d}f(x, y) = y^A \partial_A f(x, y) , \quad (3.19)$$

and,

$$\epsilon^b = \begin{cases} 1 & \text{when } b = B , \\ -1 & \text{when } b = \beta . \end{cases} \quad (3.20)$$

Following [54], we Taylor expand both sides of (3.16) around $y = 0$ and equate the n -th order terms. This gives,

$$\begin{aligned} \mathbf{d}_0^n e^{(a)}{}_B(x, 0) &= \frac{n-1}{n+1} \mathbf{d}_0^{n-2} [\rho^{(a)}{}_{(c)}(x, 0; y) e^{(c)}{}_B(x, 0)] , & n \geq 1 , \\ \mathbf{d}_0^n e^{(a)}{}_\beta(x, 0) &= \mathbf{d}_0^{n-2} [\rho^{(a)}{}_{(c)}(x, 0; y) e^{(c)}{}_\beta(x, 0)] , & n \geq 2 , \end{aligned} \quad (3.21)$$

where we have introduced a new notation,

$$\mathbf{d}_0^n [f(x, 0; y)g(x, 0)] = y^{A_1} \cdots y^{A_n} \lim_{\tilde{y} \rightarrow 0} \frac{\partial}{\partial \tilde{y}^{A_1}} \cdots \frac{\partial}{\partial \tilde{y}^{A_n}} [f(x, \tilde{y}; y)g(x, \tilde{y})] . \quad (3.22)$$

The integral forms that solve the above equations are as follows,

$$\begin{aligned} e^{(a)}{}_B(x, y) &= \delta_B^a + \int_0^1 dt F_1(t) \rho^{(a)}{}_{(c)}(x, ty; y) e^{(c)}{}_B(x, ty) , \\ e^{(a)}{}_\beta(x, y) &= \underline{e^{(a)}{}_\beta} + y^A \underline{e^{(a)}{}_{\beta, A}} + \int_0^1 dt F_3(t) \rho^{(a)}{}_{(c)}(x, ty; y) e^{(c)}{}_\beta(x, ty) . \end{aligned} \quad (3.23)$$

The first term in the first equation and the first two terms in the second equation are ‘‘initial conditions’’. The former is obtained by contracting the first equation with y^B , using (3.8) and then noticing that the integral term does not contribute under such a contraction because of the anti-symmetry property of the Riemann tensor. For the latter we have⁷,

$$\underline{e^{(\alpha)}{}_\beta} = E^{(\alpha)}{}_\beta(x) , \quad \underline{e^{(A)}{}_\beta} = 0 , \quad y^A \underline{e^{(a)}{}_{\beta, A}} = y^A \underline{\omega_\beta^{(a)}}_A , \quad (3.24)$$

⁷In the rest of the discussion in this subsection, we will refrain from explicitly writing down the arguments. It is understood that a geometric quantity denoted by a lower case symbol without a bar has an argument (x, y) and the same with a bar has an argument (x) .

where $E^{(\alpha)}_\beta$ is the vielbein of the induced metric $G_{\alpha\beta}$ on the submanifold. While the first equation is obvious, the second equation is required by consistency with (2.2). The third equation is obtained as follows. Using the following relation⁸,

$$\partial_a e^{(b)}_c - \gamma_{ac}^d e^{(b)}_d + \omega_a^{(b)}{}_{(e)} e^{(e)}_c = 0, \quad (3.25)$$

and (3.8) one gets on the submanifold,

$$\underline{\gamma_{\beta A}^d e^{(a)}_d} = \underline{\gamma_{A\beta}^d e^{(a)}_d} = \underline{\omega_\beta^{(a)}{}_A}, \quad (3.26)$$

Then using (3.25), (3.26) and (3.9) one arrives at the third equation in (3.24).

3.2 Closed form expressions for curvature expansion-coefficients

We write,

$$e^{(a)}_b(x, y) = \sum_{p=0}^{\infty} e_p^{(a)}_b(x, y), \quad (3.27)$$

where e_p is the contribution at p -th order in curvature, such that,

$$e_0^{(a)}_B = \delta_B^a, \\ e_0^{(a)}_\beta = \begin{cases} E^{(\alpha)}_\beta + \underline{\omega_\beta^{(\alpha)}{}_C} y^C, & \text{for } a = \alpha, \\ \underline{\omega_\beta^{(A)}{}_C} y^C, & \text{for } a = A. \end{cases} \quad (3.28)$$

To obtain the curvature-expansion we proceed as follows. First we use (3.27) in eqs.(3.23) iteratively to obtain, for $p \geq 1$,

$$e_p^{(a)}_b(x, y) = \int_0^1 dt_1 (t_1^{2p-2} F_q(t_1)) \int_0^1 dt_2 (t_2^{2p-4} F_q(t_2)) \cdots \int_0^1 dt_p (t_p^0 F_q(t_p)) \\ [\rho(x, t_1 y, y) \rho(x, t_1 t_2 y; y) \cdots \rho(x, t_1 \cdots t_p y; y) e_0(x, t_1 \cdots t_p y)]^{(a)}_b, \quad (3.29)$$

where $q = 1, 3$ for $b = B, \beta$ respectively. Then we Taylor expand each ρ factor (around origin of the second argument) in the above equations to get,

$$e_p^{(a)}_B(x, y) = \sum_{s_1, \dots, s_p \geq 0} \mathcal{F}_1^{(p)}(s_1, s_2, \dots, s_p) [(y \cdot \nabla)^{s_1} \rho(x, 0; y) \cdots (y \cdot \nabla)^{s_p} \rho(x, 0; y)]^{(a)}_B e^{(b)}_B,$$

⁸The LHS of eq.(3.25) is usually called the *total covariant derivative* of vielbein. The vanishing of this is sometimes referred to as *vielbein postulate*. However, this equation directly follows from (A.65) and therefore can be viewed as the defining equation for spin connection.

$$\begin{aligned}
e_p^{(a)}{}_\beta(x, y) &= \sum_{s_1, \dots, s_p \geq 0} \mathcal{F}_3^{(p)}(s_1, s_2, \dots, s_p) [(y \cdot \nabla)^{s_1} \rho(x, 0; y) \cdots (y \cdot \nabla)^{s_p} \rho(x, 0; y)]^{(a)} {}_{(b)} \underline{e}_\beta^{(b)} \\
&+ \sum_{s_1, \dots, s_p \geq 0} \mathcal{F}_1^{(p)}(s_1, s_2, \dots, s_p) [(y \cdot \nabla)^{s_1} \rho(x, 0; y) \cdots (y \cdot \nabla)^{s_p} \rho(x, 0; y)]^{(a)} {}_{(b)} \underline{\omega}_\beta^{(b)} {}_C y^C,
\end{aligned} \tag{3.30}$$

where,

$$\begin{aligned}
\mathcal{F}_q^{(p)}(s_1, s_2, \dots, s_p) &= \frac{1}{s_1! \cdots s_p!} \int_0^1 dt_1 t_1^{s_1 + \cdots + s_p + 2p - 2} F_q(t_1) \int_0^1 dt_2 t_2^{s_2 + \cdots + s_p + 2p - 4} F_q(t_2) \\
&\cdots \int_0^1 dt_p t_p^{s_p} F_q(t_p), \quad q = 1, 3,
\end{aligned} \tag{3.31}$$

and $F_1(t)$ and $F_3(t)$ are defined in eqs.(2.5). Explicit calculations give the following results,

$$\begin{aligned}
\mathcal{F}_1^{(p)}(s_1, s_2, \dots, s_p) &= \frac{C_1^{(p)}(s_1, s_2, \dots, s_p)}{(s_1 + s_2 + \cdots + s_p + 2p + 1)!}, \\
\mathcal{F}_3^{(p)}(s_1, s_2, \dots, s_p) &= \frac{C_3^{(p)}(s_1, s_2, \dots, s_p)}{(s_1 + s_2 + \cdots + s_p + 2p)!},
\end{aligned} \tag{3.32}$$

where,

$$\begin{aligned}
C_1^{(p)}(s_1, s_2, \dots, s_p) &= C_{s_1}^{s_1 + s_2 + \cdots + s_p + 2p - 1} C_{s_2}^{s_2 + s_3 + \cdots + s_p + 2p - 3} \cdots C_{s_p}^{s_p + 1}, \\
C_3^{(p)}(s_1, s_2, \dots, s_p) &= C_{s_1}^{s_1 + s_2 + \cdots + s_p + 2p - 2} C_{s_2}^{s_2 + s_3 + \cdots + s_p + 2p - 4} \cdots 1,
\end{aligned} \tag{3.33}$$

C_r^n being the binomial coefficients.

In order to present the results in convenient matrix forms we introduce $d \times d$ matrices $\underline{\mathbb{E}}^\parallel$, $\underline{\mathbb{E}}^\perp$, $\underline{\mathbb{W}}$ and $\underline{\mathbb{R}}_{2+s}(x, y)$ such that their elements are given as follows,

$$[\underline{\mathbb{E}}^\parallel]_{ab} = \begin{cases} E^{(\alpha)}{}_\beta & \text{for } a = \alpha, b = \beta \\ 0 & \text{otherwise} \end{cases}, \tag{3.34}$$

$$[\underline{\mathbb{E}}^\perp]_{ab} = \begin{cases} \delta^A{}_B & \text{for } a = A, b = B \\ 0 & \text{otherwise} \end{cases}, \tag{3.35}$$

$$[\underline{\mathbb{W}}]_{ab} = \begin{cases} \omega_\beta^{(a)} {}_C y^C & \text{for } b = \beta \\ 0 & \text{otherwise} \end{cases}, \tag{3.36}$$

$$[\underline{\mathbb{R}}_{2+s}(x, y)]_{ab} = (y \cdot \nabla)^s \rho^{(a)}{}_{(b)}(x, 0; y). \tag{3.37}$$

Notice that R_{2+s} is linear in curvature, but $(s+2)$ -th order in y . Using these we further define the following two matrices:

$$\mathbb{E}_q(x, y) = \mathbb{I} + \sum_{p=1}^{\infty} \sum_{s_1, \dots, s_p \geq 0} \mathcal{F}_q^{(p)}(s_1, \dots, s_p) R_{2+s_1}(x, y) \cdots R_{2+s_p}(x, y), \quad q = 1, 3, \quad (3.38)$$

where \mathbb{I} is the $d \times d$ identity matrix. The vielbein matrix is given by,

$$\begin{aligned} \mathbb{E}(x, y) &= \mathbb{E}^{\parallel}(x, y) + \mathbb{E}^{\perp}(x, y), \\ \mathbb{E}^{\parallel}(x, y) &= \mathbb{E}_3(x, y) \underline{\mathbb{E}}^{\parallel} + \mathbb{E}_1(x, y) \mathbb{W}, \quad \mathbb{E}^{\perp}(x, y) = \mathbb{E}_1(x, y) \underline{\mathbb{E}}^{\perp}, \end{aligned} \quad (3.39)$$

such that the nonzero elements of $\mathbb{E}^{\parallel}(x, y)$ and $\mathbb{E}^{\perp}(x, y)$ are given by $e^{(a)}_{\beta}(x, y)$ and $e^{(a)}_B(x, y)$ respectively.

The metric is given by,

$$g_{ab}(x, y) = e^{(a')}_{a}(x, y) \eta_{(a'b')} e^{(b')}_{b}(x, y). \quad (3.40)$$

In matrix form we write,

$$g(x, y) = \mathbb{E}^T(x, y) \eta \mathbb{E}(x, y) = g^{\parallel} + h + h^T + g^{\perp}, \quad (3.41)$$

where

$$\begin{aligned} g^{\parallel} &= (\mathbb{E}^{\parallel})^T \eta \mathbb{E}^{\parallel} = (\underline{\mathbb{E}}^{\parallel})^T X \underline{\mathbb{E}}^{\parallel} + (\mathbb{E}^{\parallel})^T Y \mathbb{W} + \mathbb{W}^T Y^T \underline{\mathbb{E}}^{\parallel} + \mathbb{W}^T Z \mathbb{W}, \\ h &= (\mathbb{E}^{\parallel})^T \eta \mathbb{E}^{\perp} = (\underline{\mathbb{E}}^{\parallel})^T Y \underline{\mathbb{E}}^{\perp} + \mathbb{W}^T Z \underline{\mathbb{E}}^{\perp}, \\ g^{\perp} &= (\mathbb{E}^{\perp})^T \eta \mathbb{E}^{\perp} = (\underline{\mathbb{E}}^{\perp})^T Z \underline{\mathbb{E}}^{\perp}, \end{aligned} \quad (3.42)$$

are all $d \times d$ matrices whose non-zero elements are,

$$[g^{\parallel}]_{\alpha\beta} = g_{\alpha\beta}(x, y), \quad [h]_{\alpha B} = g_{\alpha B}(x, y), \quad [g^{\perp}]_{AB} = g_{AB}(x, y). \quad (3.43)$$

Equations (3.39) and (3.42) imply,

$$X = (\bar{\mathbb{E}}_3)^T \mathbb{E}_3, \quad Y = (\bar{\mathbb{E}}_3)^T \mathbb{E}_1, \quad Z = (\bar{\mathbb{E}}_1)^T \mathbb{E}_1. \quad (3.44)$$

where,

$$\bar{\mathbb{E}}_q = \eta \mathbb{E}_q. \quad (3.45)$$

We will display explicit results for the matrices \mathbb{E}_1 , \mathbb{E}_3 , X , Y and Z up to 10-th order in y in appendix B.

4 Alternative proof of Florides-Synge theorem

Here we prove the integral theorem in (2.3) starting from our result in (3.16). Analogous integral equations for vielbein, namely (3.23), were derived from a set of equations (3.21) describing all the necessary transverse derivatives of the vielbein evaluated on the sub-manifold. We will first derive the analogue of eqs.(3.21) for the metric (eqs.(4.54) below). The integral equations in (2.3) will then follow as solutions to eqs.(4.54).

Writing $g_{ab} = e_a \cdot e_b = \eta_{(cd)} e_a^{(c)} e_b^{(d)}$ and using (3.16) we get,

$$\mathbf{d}^2 g_{ab} + \overset{a}{\epsilon} \mathbf{d} e_a \cdot e_b + \overset{b}{\epsilon} e_a \cdot \mathbf{d} e_b = 2(r_{aCDb} y^C y^D + \mathbf{d} e_a \cdot \mathbf{d} e_b) . \quad (4.46)$$

Next we show that the right hand side receives contribution only from the linear part of the curvature tensor, i.e.,

$$r_{aCDb} y^C y^D + \mathbf{d} e_a \cdot \mathbf{d} e_b = l_{aCDb} y^C y^D , \quad (4.47)$$

where l_{abcd} is as defined in eq.(2.6). To establish the above equation we first write,

$$r_{abcd} = l_{abcd} + q_{abcd} , \quad (4.48)$$

where the quadratic part of the curvature tensor is given by,

$$q_{abcd} = g_{ef} (\gamma_{ad}^e \gamma_{bc}^f - \gamma_{ac}^e \gamma_{bd}^f) . \quad (4.49)$$

The condition in (3.10) implies that the first term does not contribute in the contraction $q_{aCDb} y^C y^D$. The contribution from the second term can be calculated by using,

$$\partial_a g_{bc} y^C = (\eta_{bc} - g_{bc}) \delta_a^C , \quad (4.50)$$

which follows from eq.(2.2). The result is,

$$\begin{aligned} q_{aCDb} y^C y^D &= -\frac{1}{4} g^{cd} (\mathbf{d} g_{ca} + g_{aC} \delta_c^C - g_{cC} \delta_a^C) (\mathbf{d} g_{db} + g_{bD} \delta_d^D - g_{dD} \delta_b^D) , \\ &= -\mathbf{d} e_a \cdot \mathbf{d} e_b , \end{aligned} \quad (4.51)$$

establishing (4.47). To arrive at the second line we used the following result,

$$\begin{aligned} \mathbf{d} e_a \cdot e_b - e_a \cdot \mathbf{d} e_b &= y^C (g_{bd} \gamma_{Ca}^d - g_{ad} \gamma_{Cb}^d) , \\ &= g_{aC} \delta_b^C - g_{bC} \delta_a^C , \end{aligned} \quad (4.52)$$

where the first line is obtained by using an analogue of eq.(3.25) with indices properly placed and (3.9). The second line results from a direct calculation using the explicit form of the Christoffel symbols in terms of metric.

Using (4.47) and (4.52) in (4.46) we now rewrite the relevant second order differential equations for various metric components in the following way,

$$\begin{aligned}\mathbf{d}^2 g_{AB} + \mathbf{d}g_{AB} &= 2y^C y^D l_{ACDB} , \\ \mathbf{d}^2 g_{A\beta} - g_{A\beta} &= 2y^C y^D l_{ACD\beta} , \\ \mathbf{d}^2 g_{\alpha\beta} - \mathbf{d}g_{\alpha\beta} &= 2y^C y^D l_{\alpha CD\beta} .\end{aligned}\tag{4.53}$$

Following the method of §3.1, we Taylor expand each of the above equations around $y = 0$ to find the analogues of eqs.(3.21). The results are as follows,

$$\begin{aligned}\mathbf{d}_0^n g_{AB}(x, 0) &= 2 \frac{n-1}{n+1} y^C y^D \mathbf{d}_0^{n-2} l_{ACDB}(x, 0) , \quad n \geq 1 , \\ \mathbf{d}_0^n g_{A\beta}(x, 0) &= \frac{2n}{n+1} y^C y^D \mathbf{d}_0^{n-2} l_{ACD\beta}(x, 0) , \quad n \geq 2 , \\ \mathbf{d}_0^n g_{\alpha\beta}(x, 0) &= 2y^C y^D \mathbf{d}_0^{n-2} l_{\alpha CD\beta}(x, 0) , \quad n \geq 2 .\end{aligned}\tag{4.54}$$

The integral equations that solve the above equations are precisely the ones in eqs.(2.3).

5 An example

Here we briefly discuss a demonstrative example of our general result. In [58] Klein and Collas discussed a class of physically interesting backgrounds. For a specific worldline the exact FNC was constructed and the metric was computed in that system. Here we will demonstrate how our result reproduces the same metric up to second order in Fermi expansion.

In an *a priori* coordinate system $\bar{z}^{\bar{a}} = (z^0, z^i)$, ($i = 1, \dots, d-1$)⁹, the metric is given by,

$$ds^2 = -(1 - f(z^i))dz^0 dz^0 + \sum_{i=1}^{d-1} dz^i dz^i + [(1 - kl^2)^{-1} - 1]d\bar{l}d\bar{l} ,\tag{5.55}$$

⁹Our notation is different from [58]. We leave the total spacetime dimension d arbitrary.

where, $\bar{l}^2 = \sum_i (z^i)^2$. The local coordinate chart is defined over a region such that $(1 - k\bar{l}^2) > 0$. Furthermore, the function $f(z^i)$ satisfies the following conditions,

$$f(0) = 0, \quad \bar{\partial}_i f(0) = 0, \quad (5.56)$$

and $(1 - f(z^i))$ remains positive within the local chart. The one-dimensional submanifold under consideration is given by,

$$z^0 = x, \quad z^i = 0. \quad (5.57)$$

According to the general result of the present paper, the FNC expansion of various components of the metric are given by the following expressions up to second order,

$$\begin{aligned} g_{00} &= G_{00} + 2\underline{\omega_{00C}}y^C + \underline{(\omega_0^a{}_C \omega_{0aD} + r_{0CD0})}y^C y^D, \\ g_{0B} &= \underline{\omega_{0BC}}y^C + \frac{2}{3}\underline{r_{0CDB}}y^C y^D, \\ g_{AB} &= \eta_{AB} + \frac{1}{3}\underline{r_{ACDB}}y^C y^D, \end{aligned} \quad (5.58)$$

where $G_{00} = -1$ is the induced metric on the worldline and the signature is given by $\eta = \text{diag}(-1, 1, \dots, 1)$. Following our notation in this paper, in the above equations we have used lower case symbols to denote tensors in FNC. To denote tensors in *a priori* system we will use the same symbols with a $\bar{}$ at the top.

In order to evaluate the above expansion, one has to relate the expansion coefficients to tensors in *a priori* system which can then in turn be evaluated from the metric in (5.55) in a straightforward manner. Although the complete transformation relating $\bar{z}^{\bar{a}}$ and z^a has been found in [58], for our purpose only the Jacobian matrix evaluated at (5.57) is needed. It turns out that this is simply given by the identity matrix. Therefore, for example,

$$\underline{\omega_{abc}} = \delta_a^{\bar{a}} \delta_b^{\bar{b}} \delta_c^{\bar{c}} \bar{\omega}_{\bar{a}\bar{b}\bar{c}}. \quad (5.59)$$

The specific results that we need are as follows,

$$\begin{aligned} \underline{\bar{\omega}_{0\bar{a}k}} &= \underline{\bar{e}_{(\bar{b})\bar{a}}(-\bar{\partial}_0 \bar{e}^{(\bar{b})}{}_k + \bar{\gamma}_{0k}^{\bar{d}} \bar{e}^{(\bar{b})}{}_{\bar{d}})} = 0, \\ \underline{\bar{r}_{0jk0}} &= \frac{1}{2} \underline{\bar{\partial}_j \bar{\partial}_k f}, \quad \underline{\bar{r}_{ijk0}} = 0, \quad \underline{\bar{r}_{ijkl}} = k(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}), \end{aligned} \quad (5.60)$$

where in the first line we have used the fact that the metric in (5.55) and hence the vielbein is independent of z^0 and that $\underline{\bar{\gamma}_{0k}^{\bar{a}}} = 0$ which can easily be verified through explicit computation.

Using the above results we find,

$$\begin{aligned}
g_{00} &= -1 + \frac{1}{2} \underline{\partial_C \partial_D f} y^C y^D , \\
g_{0B} &= 0 , \\
g_{AB} &= \eta_{AB} - \frac{k}{3} (\eta_{AB} \eta_{CD} - \eta_{AC} \eta_{BD}) y^C y^D .
\end{aligned} \tag{5.61}$$

This is precisely the results in eqs.(15 - 20) in [58] expanded up to quadratic order in our notation.

6 Conclusion

The result derived in this work can be viewed as a general theorem. It says that given any tubular neighborhood, it is possible to construct a special coordinate system as described by eqs.(2.1, 2.2) such that the vielbein components in this system satisfy the integral equations as given in (3.23) and have the Taylor expansions as given in eqs.(3.27, 3.28, 3.30).

As demonstrated in §5, this can be used to compute FNC expansion of tensors when the metric and the submanifold under question are known in an arbitrary *a priori* system. The above example is particularly simple because of its special nature. It will be useful to have a completely general framework that works for an arbitrary submanifold.

One specific example is the multi-particle and string dynamics in M as mentioned in §1. Since both M^n and LM are constructed entirely using M , all the geometric properties of these two spaces are expressible in terms of the geometric data of M (M -data). This is true, for example, for the tubular expansion of the action or the Hamiltonian that describe a system with the above two spaces as configuration space. The problem at hand is therefore to express such tubular expansions in terms of M -data. Investigation to answer such questions is in progress [59].

A Notations and conventions

We denote the coordinate and non-coordinate bases of the tangent space by \mathcal{E}_a and $\hat{\mathcal{E}}_{(a)}$ respectively,

$$\mathcal{E}_b = e^{(a)}{}_b \hat{\mathcal{E}}_{(a)} , \tag{A.62}$$

$e^{(a)}_b$ being the vielbein-components. We denote by η the diagonal matrix with indicators of the non-coordinate basis as diagonal elements,

$$\eta_{ab} = \langle \hat{\mathcal{E}}_{(a)}, \hat{\mathcal{E}}_{(b)} \rangle. \quad (\text{A.63})$$

The bases dual to \mathcal{E}_a and $\hat{\mathcal{E}}_{(a)}$ are denoted by \mathcal{E}^a and $\hat{\mathcal{E}}^{(a)}$ respectively. We consider the torsion-less situation and denote the Christoffel symbols by γ^a_{bc} ,

$$\nabla_b \mathcal{E}_c \equiv \nabla_{\mathcal{E}_b} \mathcal{E}_c = \gamma^a_{bc} \mathcal{E}_a, \quad (\text{A.64})$$

where ∇ is the Levi-Civita connection. The connection one-form $\omega^{(b)}_{(c)} = \omega_{(a)(c)}^{(b)} \hat{\mathcal{E}}^{(a)}$ is given by,

$$\nabla_{(a)} \mathcal{E}_{(c)} \equiv \nabla_{\hat{\mathcal{E}}_{(a)}} \mathcal{E}_{(c)} = \omega_{(a)(c)}^{(b)} \mathcal{E}_{(b)}. \quad (\text{A.65})$$

Cartan's structure equations in our case are given by,

$$d\hat{\mathcal{E}}^{(a)} + \omega^{(a)}_{(b)} \wedge \hat{\mathcal{E}}^{(b)} = 0, \quad (\text{A.66})$$

$$d\omega^{(a)}_{(b)} + \omega^{(a)}_{(c)} \wedge \omega^{(c)}_{(b)} = r^{(a)}_{(b)}, \quad (\text{A.67})$$

where $r^{(a)}_{(b)} = \frac{1}{2} r^{(a)}_{(b)(c)(d)} \hat{\theta}^{(c)} \wedge \hat{\theta}^{(d)}$ is the curvature two-form,

$$r^{(a)}_{(b)(c)(d)} = \langle \hat{\mathcal{E}}^{(a)}, (\nabla_{(c)} \nabla_{(d)} - \nabla_{(d)} \nabla_{(c)} - \nabla_{[\hat{\mathcal{E}}_{(c)}, \hat{\mathcal{E}}_{(d)}]}) \hat{\mathcal{E}}_{(b)} \rangle, \quad (\text{A.68})$$

$[\cdot, \cdot]$ being the Lie bracket [57].

B Coefficients up to tenth order

Here we will display the expansion coefficients up to 10-th order in y for the matrices \mathbb{E}_1 , \mathbb{E}_3 , X , Y and Z . We first expand these matrices in the following way,

$$\begin{aligned} \mathbb{E}_q &= \mathbb{I} + \sum_{p=2}^{\infty} \mathbb{E}_q^{(p)}(x, y), \quad q = 1, 3, \\ X &= \eta + \sum_{p=2}^{\infty} X^{(p)} = \eta + \sum_{p=2}^{\infty} (\bar{\mathbb{E}}_3^{(p)} + \bar{\mathbb{E}}_3^{(p)T}) + \sum_{p,q=2}^{\infty} \bar{\mathbb{E}}_3^{(p)T} \mathbb{E}_3^{(q)}, \\ Y &= \eta + \sum_{p=2}^{\infty} Y^{(p)} = \eta + \sum_{p=2}^{\infty} (\bar{\mathbb{E}}_1^{(p)} + \bar{\mathbb{E}}_3^{(p)T}) + \sum_{p,q=2}^{\infty} \bar{\mathbb{E}}_3^{(p)T} \mathbb{E}_1^{(q)}, \end{aligned}$$

$$Z = \eta + \sum_{p=2}^{\infty} Z^{(p)} = \eta + \sum_{p=2}^{\infty} (\bar{\mathbb{E}}_1^{(p)} + \bar{\mathbb{E}}_1^{(p)T}) + \sum_{p,q=2}^{\infty} \bar{\mathbb{E}}_1^{(p)T} \mathbb{E}_1^{(q)}, \quad (\text{B.69})$$

where $\mathbb{E}_q^{(p)}(x, y)$, $X^{(p)}(x, y)$, $Y^{(p)}(x, y)$ and $Z^{(p)}(x, y)$ are contributions to the relevant matrices at the p -th order in y . The results up to $p = 10$ are given below. The following notation will be used in writing down results for X , Y and Z . Given a matrix of the form: $\bar{R}_m R_n \dots = \eta R_m R_n \dots$, we define,

$$\{\bar{R}_m R_n \dots\} = \eta R_m R_n \dots + \eta \dots R_n R_m. \quad (\text{B.70})$$

Results for \mathbb{E}_1 :

$$\begin{aligned} \mathbb{E}_1^{(2)} &= \frac{1}{6} R_2, \\ \mathbb{E}_1^{(3)} &= \frac{1}{12} R_3, \\ \mathbb{E}_1^{(4)} &= \frac{1}{40} R_4 + \frac{1}{120} R_2^2, \\ \mathbb{E}_1^{(5)} &= \frac{1}{180} R_5 + \frac{1}{180} R_3 R_2 + \frac{1}{360} R_2 R_3, \\ \mathbb{E}_1^{(6)} &= \frac{1}{1008} R_6 + \frac{1}{504} R_4 R_2 + \frac{1}{1680} R_2 R_4 + \frac{1}{504} R_3^2 + \frac{1}{5040} R_2^3, \\ \mathbb{E}_1^{(7)} &= \frac{1}{6720} R_7 + \frac{1}{2016} R_5 R_2 + \frac{1}{1344} R_4 R_3 + \frac{1}{2240} R_3 R_4 + \frac{1}{6720} R_3 R_2^2 + \frac{1}{10080} R_2 R_5 \\ &\quad + \frac{1}{10080} R_2 R_3 R_2 + \frac{1}{20160} R_2^2 R_3, \\ \mathbb{E}_1^{(8)} &= \frac{1}{51840} R_8 + \frac{1}{10368} R_6 R_2 + \frac{1}{5184} R_5 R_3 + \frac{1}{5760} R_4^2 + \frac{1}{17280} R_4 R_2^2 \\ &\quad + \frac{1}{12960} R_3 R_5 + \frac{1}{12960} R_3^2 R_2 + \frac{1}{25920} R_3 R_2 R_3 + \frac{1}{72576} R_2 R_6 \\ &\quad + \frac{1}{36288} R_2 R_4 R_2 + \frac{1}{36288} R_2 R_3^2 + \frac{1}{120960} R_2^2 R_4 + \frac{1}{362880} R_2^4, \\ \mathbb{E}_1^{(9)} &= \frac{1}{453600} R_9 + \frac{1}{64800} R_7 R_2 + \frac{1}{25920} R_6 R_3 + \frac{1}{21600} R_5 R_4 + \frac{1}{64800} R_5 R_2^2 + \frac{1}{32400} R_4 R_5 \\ &\quad + \frac{1}{32400} R_4 R_3 R_2 + \frac{1}{64800} R_4 R_2 R_3 + \frac{1}{90720} R_3 R_6 + \frac{1}{45360} R_3 R_4 R_2 + \frac{1}{45360} R_3^3 \\ &\quad + \frac{1}{151200} R_3 R_2 R_4 + \frac{1}{453600} R_3 R_2^3 + \frac{1}{604800} R_2 R_7 + \frac{1}{181440} R_2 R_5 R_2 + \frac{1}{120960} R_2 R_4 R_3 \\ &\quad + \frac{1}{201600} R_2 R_3 R_4 + \frac{1}{604800} R_2 R_3 R_2^2 + \frac{1}{907200} R_2^2 R_5 + \frac{1}{907200} R_2^2 R_3 R_2 \\ &\quad + \frac{1}{1814400} R_2^3 R_3, \\ \mathbb{E}_1^{(10)} &= \frac{1}{4435200} R_{10} + \frac{1}{475200} R_8 R_2 + \frac{1}{158400} R_7 R_3 + \frac{1}{105600} R_6 R_4 + \frac{1}{316800} R_6 R_2^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{118800} R_5^2 + \frac{1}{118800} R_5 R_3 R_2 + \frac{1}{237600} R_5 R_2 R_3 + \frac{1}{221760} R_4 R_6 \\
& + \frac{1}{110880} R_4^2 R_2 + \frac{1}{110880} R_4 R_3^2 + \frac{1}{369600} R_4 R_2 R_4 + \frac{1}{1108800} R_4 R_2^3 + \frac{1}{739200} R_3 R_7 \\
& + \frac{1}{221760} R_3 R_5 R_2 + \frac{1}{147840} R_3 R_4 R_3 + \frac{1}{246400} R_3^2 R_4 + \frac{1}{739200} R_3^2 R_2^2 + \frac{1}{1108800} R_3 R_2 R_5 \\
& + \frac{1}{1108800} R_3 R_2 R_3 R_2 + \frac{1}{2217600} R_3 R_2^2 R_3 + \frac{1}{5702400} R_2 R_8 \\
& + \frac{1}{1140480} R_2 R_6 R_2 + \frac{1}{570240} R_2 R_5 R_3 + \frac{1}{633600} R_2 R_4^2 \\
& + \frac{1}{1900800} R_2 R_4 R_2^2 + \frac{1}{1425600} R_2 R_3 R_5 + \frac{1}{1425600} R_2 R_3^2 R_2 \\
& + \frac{1}{2851200} R_2 R_3 R_2 R_3 + \frac{1}{7983360} R_2^2 R_6 + \frac{1}{3991680} R_2^2 R_4 R_2 \\
& + \frac{1}{3991680} R_2^2 R_3^2 + \frac{1}{13305600} R_2^3 R_4 + \frac{1}{39916800} R_2^5 .
\end{aligned}$$

Results for \mathbb{E}_3 :

$$\begin{aligned}
\mathbb{E}_3^{(2)} &= \frac{1}{2} R_2 , \\
\mathbb{E}_3^{(3)} &= \frac{1}{6} R_3 , \\
\mathbb{E}_3^{(4)} &= \frac{1}{24} R_4 + \frac{1}{24} R_2^2 , \\
\mathbb{E}_3^{(5)} &= \frac{1}{120} R_5 + \frac{1}{40} R_3 R_2 + \frac{1}{120} R_2 R_3 , \\
\mathbb{E}_3^{(6)} &= \frac{1}{720} R_6 + \frac{1}{120} R_4 R_2 + \frac{1}{720} R_2 R_4 + \frac{1}{180} R_3^2 + \frac{1}{720} R_2^3 , \\
\mathbb{E}_3^{(7)} &= \frac{1}{5040} R_7 + \frac{1}{504} R_5 R_2 + \frac{1}{504} R_4 R_3 + \frac{1}{1008} R_3 R_4 + \frac{1}{1008} R_3 R_2^2 \\
& + \frac{1}{5040} R_2 R_5 + \frac{1}{1680} R_2 R_3 R_2 + \frac{1}{5040} R_2^2 R_3 , \\
\mathbb{E}_3^{(8)} &= \frac{1}{40320} R_8 + \frac{1}{2688} R_6 R_2 + \frac{1}{2016} R_5 R_3 + \frac{1}{2688} R_4^2 + \frac{1}{2688} R_4 R_2^2 \\
& + \frac{1}{6720} R_3 R_5 + \frac{1}{2240} R_3^2 R_2 + \frac{1}{6720} R_3 R_2 R_3 + \frac{1}{40320} R_2 R_6 \\
& + \frac{1}{6720} R_2 R_4 R_2 + \frac{1}{10080} R_2 R_3^2 + \frac{1}{40320} R_2^2 R_4 + \frac{1}{40320} R_2^4 , \\
\mathbb{E}_3^{(9)} &= \frac{1}{362880} R_9 + \frac{1}{17280} R_7 R_2 + \frac{1}{10368} R_6 R_3 + \frac{1}{10368} R_5 R_4 + \frac{1}{10368} R_5 R_2^2 \\
& + \frac{1}{17280} R_4 R_5 + \frac{1}{5760} R_4 R_3 R_2 + \frac{1}{17280} R_4 R_2 R_3 + \frac{1}{51840} R_3 R_6 \\
& + \frac{1}{8640} R_3 R_4 R_2 + \frac{1}{12960} R_3^3 + \frac{1}{51840} R_3 R_2 R_4 + \frac{1}{51840} R_3 R_2^3 \\
& + \frac{1}{362880} R_2 R_7 + \frac{1}{36288} R_2 R_5 R_2 + \frac{1}{36288} R_2 R_4 R_3 + \frac{1}{72576} R_2 R_3 R_4
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{72576} R_2 R_3 R_2^2 + \frac{1}{362880} R_2^2 R_5 + \frac{1}{120960} R_2^2 R_3 R_2 \\
& + \frac{1}{362880} R_2^3 R_3 , \\
\mathbb{E}_3^{(10)} = & \frac{1}{3628800} R_{10} + \frac{1}{129600} R_8 R_2 + \frac{1}{64800} R_7 R_3 + \frac{1}{51840} R_6 R_4 + \frac{1}{51840} R_6 R_2^2 \\
& + \frac{1}{64800} R_5^2 + \frac{1}{21600} R_5 R_3 R_2 + \frac{1}{64800} R_5 R_2 R_3 + \frac{1}{129600} R_4 R_6 \\
& + \frac{1}{21600} R_4^2 R_2 + \frac{1}{32400} R_4 R_3^2 + \frac{1}{129600} R_4 R_2 R_4 \\
& + \frac{1}{129600} R_4 R_2^3 + \frac{1}{453600} R_3 R_7 + \frac{1}{45360} R_3 R_5 R_2 \\
& + \frac{1}{45360} R_3 R_4 R_3 + \frac{1}{90720} R_3^2 R_4 + \frac{1}{90720} R_3^2 R_2^2 \\
& + \frac{1}{453600} R_3 R_2 R_5 + \frac{1}{151200} R_3 R_2 R_3 R_2 + \frac{1}{453600} R_3 R_2^2 R_3 \\
& + \frac{1}{3628800} R_2 R_8 + \frac{1}{241920} R_2 R_6 R_2 + \frac{1}{181440} R_2 R_5 R_3 \\
& + \frac{1}{241920} R_2 R_4^2 + \frac{1}{241920} R_2 R_4 R_2^2 + \frac{1}{604800} R_2 R_3 R_5 \\
& + \frac{1}{201600} R_2 R_3^2 R_2 + \frac{1}{604800} R_2 R_3 R_2 R_3 + \frac{1}{3628800} R_2^2 R_6 \\
& + \frac{1}{604800} R_2^2 R_4 R_2 + \frac{1}{907200} R_2^2 R_3^2 + \frac{1}{3628800} R_2^3 R_4 \\
& + \frac{1}{3628800} R_2^5 .
\end{aligned}$$

Results for G^{\parallel} :

$$\begin{aligned}
X^{(2)} &= \bar{R}_2 , \\
X^{(3)} &= \frac{1}{3} \bar{R}_3 , \\
X^{(4)} &= \frac{1}{12} \bar{R}_4 + \frac{1}{3} \bar{R}_2 R_2 , \\
X^{(5)} &= \frac{1}{60} \bar{R}_5 + \frac{1}{60} \{\bar{R}_3 R_2\} , \\
X^{(6)} &= \frac{1}{360} \bar{R}_6 + \frac{11}{360} \{\bar{R}_4 R_2\} + \frac{7}{180} \bar{R}_3 R_3 + \frac{2}{45} \bar{R}_2 R_2^2 , \\
X^{(7)} &= \frac{1}{2520} \bar{R}_7 + \frac{2}{315} \{\bar{R}_5 R_2\} + \frac{1}{504} \{\bar{R}_4 R_3\} + \frac{31}{2520} \{\bar{R}_3 R_2^2\} + \frac{11}{420} \bar{R}_2 R_3 R_2 , \\
X^{(8)} &= \frac{1}{20160} \bar{R}_8 + \frac{11}{10080} \{\bar{R}_6 R_2\} + \frac{41}{20160} \{\bar{R}_5 R_3\} + \frac{5}{2016} \bar{R}_4 R_4 + \frac{19}{6720} \{\bar{R}_4 R_2^2\} \\
& + \frac{41}{20160} \bar{R}_3 R_5 + \frac{151}{20160} \{\bar{R}_3 R_3 R_2\} + \frac{31}{10080} \bar{R}_3 R_2 R_3 + \frac{29}{3360} \bar{R}_2 R_4 R_2 + \frac{1}{315} \bar{R}_2 R_2^3 , \\
X^{(9)} &= \frac{1}{181440} \bar{R}_9 + \frac{29}{181440} \{\bar{R}_7 R_2\} + \frac{1}{2880} \{\bar{R}_6 R_3\} + \frac{13}{25920} \{\bar{R}_5 R_4\} + \frac{11}{20160} \{\bar{R}_5 R_2^2\}
\end{aligned}$$

$$\begin{aligned}
X^{(10)} = & \frac{313}{181440} \{\bar{R}_4 R_3 R_2\} + \frac{17}{25920} \{\bar{R}_4 R_2 R_3\} + \frac{229}{90720} \{\bar{R}_3 R_4 R_2\} + \frac{13}{6480} \bar{R}_3 R_3^2 \\
& + \frac{127}{181440} \{\bar{R}_3 R_2^3\} + \frac{37}{18144} \bar{R}_2 R_5 R_2 + \frac{337}{181440} \{\bar{R}_2 R_3 R_2^2\} \\
& + \frac{1}{1814400} \bar{R}_{10} + \frac{37}{1814400} \{\bar{R}_8 R_2\} + \frac{23}{453600} \{\bar{R}_7 R_3\} + \frac{11}{129600} \{\bar{R}_6 R_4\} \\
& + \frac{163}{1814400} \{\bar{R}_6 R_2^2\} + \frac{13}{129600} \bar{R}_5 R_5 + \frac{1}{3024} \{\bar{R}_5 R_3 R_2\} + \frac{109}{907200} \{\bar{R}_5 R_2 R_3\} \\
& + \frac{353}{604800} \{\bar{R}_4 R_4 R_2\} + \frac{199}{453600} \{\bar{R}_4 R_3^2\} + \frac{17}{129600} \bar{R}_4 R_2 R_4 + \frac{247}{1814400} \{\bar{R}_4 R_2^3\} \\
& + \frac{11}{18144} \{\bar{R}_3 R_5 R_2\} + \frac{2}{2835} \bar{R}_3 R_4 R_3 + \frac{13}{28350} \{\bar{R}_3 R_3 R_2^2\} + \frac{59}{151200} \bar{R}_3 R_2 R_3 R_2 \\
& + \frac{127}{907200} \bar{R}_3 R_2^2 R_3 + \frac{23}{60480} \bar{R}_2 R_6 R_2 + \frac{53}{86400} \{\bar{R}_2 R_4 R_2^2\} + \frac{109}{100800} \bar{R}_2 R_3^2 R_2 \\
& + \frac{59}{151200} \bar{R}_2 R_3 R_2 R_3 + \frac{2}{14175} \bar{R}_2 R_2^4 .
\end{aligned}$$

Results for H :

$$\begin{aligned}
Y^{(2)} &= \frac{2}{3} \bar{R}_2 , \\
Y^{(3)} &= \frac{1}{4} \bar{R}_3 , \\
Y^{(4)} &= \frac{1}{15} \bar{R}_4 + \frac{2}{15} \bar{R}_2 R_2 , \\
Y^{(5)} &= \frac{1}{72} \bar{R}_5 + \frac{1}{24} \bar{R}_3 R_2 + \frac{5}{72} \bar{R}_2 R_3 , \\
Y^{(6)} &= \frac{1}{420} \bar{R}_6 + \frac{1}{63} \bar{R}_4 R_2 + \frac{3}{140} \bar{R}_3 R_3 + \frac{3}{140} \bar{R}_2 R_4 + \frac{5}{504} \bar{R}_2 R_2^2 , \\
Y^{(7)} &= \frac{1}{2880} \bar{R}_7 + \frac{1}{480} \bar{R}_5 R_2 + \frac{1}{192} \bar{R}_4 R_3 + \frac{19}{2880} \bar{R}_3 R_4 + \frac{1}{320} \bar{R}_3 R_2^2 + \frac{7}{1440} \bar{R}_2 R_5 \\
&+ \frac{11}{1440} \bar{R}_2 R_3 R_2 + \frac{17}{2880} \bar{R}_2 R_2 R_3 , \\
Y^{(8)} &= \frac{1}{22680} \bar{R}_8 + \frac{1}{2835} \bar{R}_6 R_2 + \frac{47}{45360} \bar{R}_5 R_3 + \frac{1}{630} \bar{R}_4 R_4 + \frac{1}{1512} \bar{R}_4 R_2^2 \\
&+ \frac{11340}{23} \bar{R}_3 R_5 + \frac{11340}{13} \bar{R}_3 R_3 R_2 + \frac{61}{45360} \bar{R}_3 R_2 R_3 + \frac{1}{1134} \bar{R}_2 R_6 + \frac{29}{11340} \bar{R}_2 R_4 R_2 \\
&+ \frac{6480}{23} \bar{R}_2 R_3^2 + \frac{13}{7560} \bar{R}_2 R_2 R_4 + \frac{2}{2835} \bar{R}_2 R_2^3 , \\
Y^{(9)} &= \frac{1}{201600} \bar{R}_9 + \frac{31}{604800} \bar{R}_7 R_2 + \frac{1}{5760} \bar{R}_6 R_3 + \frac{1}{3200} \bar{R}_5 R_4 + \frac{73}{604800} \bar{R}_5 R_2^2 \\
&+ \frac{86400}{31} \bar{R}_4 R_5 + \frac{89}{201600} \bar{R}_4 R_3 R_2 + \frac{23}{86400} \bar{R}_4 R_2 R_3 + \frac{11}{40320} \bar{R}_3 R_6 + \frac{43}{60480} \bar{R}_3 R_4 R_2 \\
&+ \frac{1}{1120} \bar{R}_3 R_3^2 + \frac{1}{2688} \bar{R}_3 R_2 R_4 + \frac{17}{120960} \bar{R}_3 R_2^3 + \frac{3}{22400} \bar{R}_2 R_7 + \frac{37}{60480} \bar{R}_2 R_5 R_2 \\
&+ \frac{1}{840} \bar{R}_2 R_4 R_3 + \frac{23}{22400} \bar{R}_2 R_3 R_4 + \frac{79}{201600} \bar{R}_2 R_3 R_2^2 + \frac{229}{604800} \bar{R}_2 R_2 R_5
\end{aligned}$$

$$\begin{aligned}
& + \frac{31}{67200} \bar{R}_2 R_2 R_3 R_2 + \frac{167}{604800} \bar{R}_2 R_2^2 R_3, \\
Y^{(10)} = & \frac{1}{1995840} \bar{R}_{10} + \frac{13}{1995840} \bar{R}_8 R_2 + \frac{5}{199584} \bar{R}_7 R_3 + \frac{37}{712800} \bar{R}_6 R_4 + \frac{191}{9979200} \bar{R}_6 R_2^2 \\
& + \frac{1}{14256} \bar{R}_5 R_5 + \frac{1}{12320} \bar{R}_5 R_3 R_2 + \frac{23}{498960} \bar{R}_5 R_2 R_3 + \frac{13}{199584} \bar{R}_4 R_6 + \frac{1}{6336} \bar{R}_4 R_4 R_2 \\
& + \frac{37}{199584} \bar{R}_4 R_3^2 + \frac{349}{4989600} \bar{R}_4 R_2 R_4 + \frac{251}{9979200} \bar{R}_4 R_2^3 + \frac{83}{1995840} \bar{R}_3 R_7 \\
& + \frac{5}{28512} \bar{R}_3 R_5 R_2 + \frac{127}{399168} \bar{R}_3 R_4 R_3 + \frac{2477}{9979200} \bar{R}_3 R_3 R_4 + \frac{899}{9979200} \bar{R}_3 R_3 R_2^2 \\
& + \frac{1}{997920} \bar{R}_3 R_2 R_5 + \frac{11088}{19} \bar{R}_3 R_2 R_3 R_2 + \frac{101}{1995840} \bar{R}_3 R_2^2 R_3 + \frac{1}{57024} \bar{R}_2 R_8 \\
& + \frac{1}{199584} \bar{R}_2 R_6 R_2 + \frac{66528}{257} \bar{R}_2 R_5 R_3 + \frac{163}{475200} \bar{R}_2 R_4^2 + \frac{139}{1108800} \bar{R}_2 R_4 R_2^2 \\
& + \frac{1}{4455} \bar{R}_2 R_3 R_5 + \frac{66528}{997920} \bar{R}_2 R_3^2 R_2 + \frac{29}{199584} \bar{R}_2 R_3 R_2 R_3 + \frac{1}{14784} \bar{R}_2 R_2 R_6 \\
& + \frac{65}{399168} \bar{R}_2 R_2 R_4 R_2 + \frac{19}{99792} \bar{R}_2 R_2 R_3^2 + \frac{713}{9979200} \bar{R}_2 R_2^2 R_4 + \frac{4}{155925} \bar{R}_2 R_2^4,
\end{aligned}$$

Results for G^\perp :

$$\begin{aligned}
Z^{(2)} &= \frac{1}{3} \bar{R}_2, \\
Z^{(3)} &= \frac{1}{6} \bar{R}_3, \\
Z^{(4)} &= \frac{1}{20} \bar{R}_4 + \frac{2}{45} \bar{R}_2 R_2, \\
Z^{(5)} &= \frac{1}{90} \bar{R}_5 + \frac{1}{45} \{\bar{R}_3 R_2\}, \\
Z^{(6)} &= \frac{1}{504} \bar{R}_6 + \frac{17}{2520} \{\bar{R}_4 R_2\} + \frac{11}{1008} \bar{R}_3 R_3 + \frac{1}{315} \bar{R}_2^3, \\
Z^{(7)} &= \frac{1}{3360} \bar{R}_7 + \frac{23}{15120} \{\bar{R}_5 R_2\} + \frac{11}{3360} \{\bar{R}_4 R_3\} + \frac{41}{30240} \{\bar{R}_3 R_2^2\} + \frac{31}{15120} \bar{R}_2 R_3 R_2 \\
Z^{(8)} &= \frac{1}{25920} \bar{R}_8 + \frac{5}{18144} \{\bar{R}_6 R_2\} + \frac{19}{25920} \{\bar{R}_5 R_3\} + \frac{7}{7200} \bar{R}_4 R_4 + \frac{113}{302400} \{\bar{R}_4 R_2^2\} \\
& + \frac{163}{181440} \{\bar{R}_3 R_3 R_2\} + \frac{7}{12960} \bar{R}_3 R_2 R_3 + \frac{13}{18144} \bar{R}_2 R_4 R_2 + \frac{2}{14175} \bar{R}_2 R_2^3, \\
Z^{(9)} &= \frac{1}{226800} \bar{R}_9 + \frac{19}{453600} \{\bar{R}_7 R_2\} + \frac{1}{7560} \{\bar{R}_6 R_3\} + \frac{7}{32400} \{\bar{R}_5 R_4\} \\
& + \frac{1}{12600} \{\bar{R}_5 R_2^2\} + \frac{113}{453600} \bar{R}_4 R_3 R_2 + \frac{2}{14175} \bar{R}_4 R_2 R_3 + \frac{29}{90720} \bar{R}_3 R_4 R_2 \\
& + \frac{17}{45360} \bar{R}_3 R_3^2 + \frac{2}{14175} \bar{R}_3 R_2 R_4 + \frac{23}{453600} \{\bar{R}_3 R_2^3\} + \frac{1}{5670} \bar{R}_2 R_5 R_2 \\
& + \frac{29}{90720} \bar{R}_2 R_4 R_3 + \frac{113}{453600} \bar{R}_2 R_3 R_4 + \frac{41}{453600} \{\bar{R}_2 R_3 R_2^2\}, \\
Z^{(10)} &= \frac{1}{2217600} \bar{R}_{10} + \frac{47}{8553600} \{\bar{R}_8 R_2\} + \frac{89}{4435200} \{\bar{R}_7 R_3\} + \frac{43}{1108800} \{\bar{R}_6 R_4\} + \frac{829}{59875200} \{\bar{R}_6 R_2^2\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{17}{356400} \bar{R}_5 R_5 + \frac{113}{2138400} \bar{R}_5 R_3 R_2 + \frac{23}{798336} \bar{R}_5 R_2 R_3 + \frac{593}{6652800} \{\bar{R}_4 R_4 R_2\} \\
& + \frac{443}{4435200} \{\bar{R}_4 R_3^2\} + \frac{13}{369600} \bar{R}_4 R_2 R_4 + \frac{7}{570240} \{\bar{R}_4 R_2^3\} \\
& + \frac{191}{2395008} \bar{R}_3 R_5 R_2 + \frac{61}{443520} \bar{R}_3 R_4 R_3 + \frac{601}{17107200} \{\bar{R}_3 R_3 R_2^2\} + \frac{23}{798336} \bar{R}_3 R_2 R_5 \\
& + \frac{1879}{59875200} \bar{R}_3 R_2 R_3 R_2 + \frac{337}{19958400} \bar{R}_3 R_2^2 R_3 + \frac{29}{855360} \bar{R}_2 R_6 R_2 + \frac{191}{2395008} \bar{R}_2 R_5 R_3 \\
& + \frac{1889}{59875200} \bar{R}_2 R_4 R_2^2 + \frac{113}{2138400} \bar{R}_2 R_3 R_5 + \frac{31}{534600} \bar{R}_2 R_3^2 R_2 + \frac{1879}{59875200} \bar{R}_2 R_3 R_2 R_3 \\
& + \frac{1889}{59875200} \bar{R}_2 R_2 R_4 R_2 + \frac{2}{467775} \bar{R}_2 R_2^4 .
\end{aligned}$$

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