Bianchi type-V dark energy model with varying EoS parameter

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Within the scope of an anisotropic Bianchi type-V cosmological model we have studied the evolution of the universe. The assumption of a diagonal energy-momentum tensor leads to some severe restriction on the metric functions, which on its part imposes restriction on the components of the energy momentum tensor. This model allows anisotropic matter distribution. Further using the proportionality condition that relates the shear scalar (σ) in the model is proportional to expansion scalar (ϑ) and the variation law of Hubble parameter, connecting Hubble parameter with volume scale. Exact solution to the corresponding equations are obtained. The EoS parameter for dark energy as well as deceleration parameter is found to be the time varying functions. A qualitative picture of the evolution of the universe corresponding to different of its stages is given using the latest observational data.

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I. INTRODUCTION

The discovery of late time accelerating mode of expansion of the Universe in one hand gave a boost to observational cosmology, at the same time posing new challenges to cosmologists. Since its discovery a number of models are offered to explain this phenomenon. Most of the dark energy models such as quintessence, Chaplygin gas etc. are simulated in analogy with the cosmological constant that gives rise to a negative pressure. In doing so a constant EoS parameter was considered. Recently in a number of papers different cosmological models with time dependent EoS parameter was studied [1, 2, 16, 17, 20, 21, 30]. The aim of the current paper is to extend that study for a Bianchi type-V cosmological model. It should be noted that a BV model can be deduced from a BVI with some suitable choice of spatial dependence of the metric function. A Bianchi type-V model describes an anisotropic but homogeneous Universe. This model was studied by several authors [8, 18, 19, 22, 23, 27], specially due to the existence of magnetic fields in galaxies which was proved by a number of astrophysical observations. Whereas, some dark energy model within the scope of a BV cosmology was studied in [29].

II. BASIC EQUATIONS

Bianchi type-V model given be given by [18, 19]

$$ds^2 = dt^2 - a_1^2 e^{-2mz} dx^2 - a_2^2 e^{-2mz} dy^2 - a_3^2 dz^2, \tag{2.1}$$

with a_1, a_2, a_3 being the functions of time only. Here m is some arbitrary constants and the velocity of light is taken to be unity. Here we consider the case when the energy momentum tensor has only non-trivial diagonal elements, i.e.

$$T_\alpha^\beta = \text{diag}[T_0^0, T_1^1, T_2^2, T_3^3] \tag{2.2}$$

Einstein field equations for the metric (2.1) on account of (2.2) have the form [18]

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} - \frac{m^2}{a_3^2} = \kappa T_1^1, \tag{2.3a}$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} - \frac{m^2}{a_3^2} = \kappa T_2^2, \tag{2.3b}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} - \frac{m^2}{a_3^2} = \kappa T_3^3, \tag{2.3c}$$

$$\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} - 3\frac{m^2}{a_3^2} = \kappa T_0^0, \tag{2.3d}$$

$$\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} - 2\frac{\dot{a}_3}{a_3} = 0. \tag{2.3e}$$

We define the spatial volume of the model (2.1) as

$$V = a_1 a_2 a_3, \tag{2.4}$$

and the average scale factor as

$$a = V^{1/3} = (a_1 a_2 a_3)^{1/3}. \tag{2.5}$$

Let us now find expansion and shear for BVI metric. The expansion is given by

$$\vartheta = u^\mu_{;\mu} = u^\mu_\mu + \Gamma^\mu_{\mu\alpha} u^\alpha, \tag{2.6}$$

and the shear is given by

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \tag{2.7}$$

with

$$\sigma_{\mu\nu} = \frac{1}{2}\left[u_{\mu;\alpha}P_{\nu}^{\alpha} + u_{\nu;\alpha}P_{\mu}^{\alpha}\right] - \frac{1}{3}\vartheta P_{\mu\nu}, \tag{2.8}$$

where the projection vector P:

$$P^2 = P, \quad P_{\mu\nu} = g_{\mu\nu} - u_{\mu}u_{\nu}, \quad P_{\nu}^{\mu} = \delta_{\nu}^{\mu} - u^{\mu}u_{\nu}. \tag{2.9}$$

In comoving system we have $u^{\mu} = (1,0,0,0)$. In this case one finds

$$\vartheta = \frac{\dot{a_1}}{a_1} + \frac{\dot{a_2}}{a_2} + \frac{\dot{a_3}}{a_3} = \frac{\dot{V}}{V}, \tag{2.10}$$

and

$$\sigma_1^1 = \frac{1}{3}\left(-2\frac{\dot{a_1}}{a_1} + \frac{\dot{a_2}}{a_2} + \frac{\dot{a_3}}{a_3}\right) = \frac{\dot{a_1}}{a_1} - \frac{1}{3}\vartheta, \tag{2.11}$$

$$\sigma_2^2 = \frac{1}{3}\left(-2\frac{\dot{a_2}}{a_2} + \frac{\dot{a_3}}{a_3} + \frac{\dot{a_1}}{a_1}\right) = \frac{\dot{a_2}}{a_2} - \frac{1}{3}\vartheta, \tag{2.12}$$

$$\sigma_3^3 = \frac{1}{3}\left(-2\frac{\dot{a_3}}{a_3} + \frac{\dot{a_1}}{a_1} + \frac{\dot{a_2}}{a_2}\right) = \frac{\dot{a_3}}{a_3} - \frac{1}{3}\vartheta. \tag{2.13}$$

One then finds

$$\sigma^2 = \frac{1}{2}\left[\sum_{i=1}^{3}\left(\frac{\dot{a_i}}{a_i}\right)^2 - \frac{1}{3}\vartheta^2\right] = \frac{1}{2}\left[\sum_{i=1}^{3}H_i^2 - \frac{1}{3}\vartheta^2\right]. \tag{2.14}$$

The Hubble constant of the model is defined by

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{a_1}}{a_1} + \frac{\dot{a_2}}{a_2} + \frac{\dot{a_3}}{a_3}\right) = \frac{1}{3}\frac{\dot{V}}{V}. \tag{2.15}$$

The deceleration parameter q, and the average anisotropy parameter A_m are defined by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = 2 - 3\frac{V\ddot{V}}{\dot{V}^2}, \tag{2.16}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i}{H} - 1\right)^2, \tag{2.17}$$

where H_i are the directional Hubble constants:

$$H_1 = \frac{\dot{a_1}}{a_1}, \quad H_2 = \frac{\dot{a_2}}{a_2}, \quad H_3 = \frac{\dot{a_3}}{a_3}. \tag{2.18}$$

III. SOLUTION TO THE FIELD EQUATIONS

From (2.3e) immediately follows

$$a_1 a_2 = k_1 a_3^2, \quad k_1 = \text{const.} \tag{3.1}$$

We also impose use the proportionality condition, widely used in literature. Demanding that the expansion is proportion to a component of the shear tensor, namely

$$\vartheta = N_1 \sigma_1^1. \tag{3.2}$$

The motivation behind assuming this condition is explained with reference to Thorne [25], the observations of the velocity-red-shift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic today within ≈ 30 per cent [10, 11]. To put more precisely, red-shift studies place the limit

$$\frac{\sigma}{H} \le 0.3, \tag{3.3}$$

on the ratio of shear σ to Hubble constant H in the neighborhood of our Galaxy today. Collins et al. (1980) have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfies that the condition $\frac{\sigma}{\theta}$ is constant.

On account of (2.10) and (2.13) we find

$$a_1 = N_0 V^{\frac{1}{3} + \frac{1}{N_1}}, \quad N_0 = \text{const.} \tag{3.4}$$

In view of (2.4) and (3.6) from (3.1) we find

$$a_2 = \frac{k_1^{1/3}}{N_0} V^{\frac{1}{3} - \frac{1}{3N_1}}, \tag{3.5}$$

$$a_3 = \frac{1}{k_1^{1/3}} V^{\frac{1}{3}}. \tag{3.6}$$

Thus, we have derived metric functions in terms of V. In order to find the equation for V we take the following steps. Subtractions of (2.3a) from (2.3b), (2.3c) from (2.3c), and (2.3c) from (2.3a) on account of (3.4), (3.5) and (3.6) give

$$\frac{\ddot{V}}{V} = \frac{\kappa N_1}{2} [T_2^2 - T_1^1], \tag{3.7a}$$

$$\frac{\ddot{V}}{V} = -\kappa N_1 [T_3^3 - T_2^2], \tag{3.7b}$$

$$\frac{\ddot{V}}{V} = -\kappa N_1 [T_1^1 - T_3^3]. \tag{3.7c}$$

$$\tag{3.7d}$$

From (3.7) immediately follows

$$\frac{1}{2} [T_2^2 - T_1^1] == [T_3^3 - T_2^2] == [T_1^1 - T_3^3]. \tag{3.8}$$

After a little manipulation, it could be established that

$$T_1^1 + T_2^2 = 2T_3^3. \tag{3.9}$$

Hence, the energy momentum tensor can be taken as

$$
\begin{aligned}
T_\alpha^\beta &= \text{diag}[\varepsilon, -p_x, -p_y, -p_z], \\
&= \text{diag}[1, -\omega_x, -\omega_y, -\omega_z]\varepsilon, \\
&= \text{diag}[1, -(\omega+\delta), -(\omega-\delta), -\omega]\varepsilon.
\end{aligned}
\tag{3.10}
$$

Thus we conclude that under the proportionality condition, the energy-momentum distribution of the model should obey (3.8).

As one sees, in order to find V we have to impose some additional condition. Let us apply the law of variation for Hubble parameter given by [3] which yields a constant value of deceleration parameter. Here, the law reads as

$$
H = Da^{-n} = DV^{-n/3},
\tag{3.11}
$$

where $D > 0$ and $n \geq 0$ are constants. Such type of relations have firstly been considered by [3, 4] for solving FRW models. Latter on many authors have used this law to study FRW and Bianchi type models. In view of (2.15) and (3.11) we find

$$
\frac{\dot{V}}{V} = 3DV^{-n/3}
\tag{3.12}
$$

with the solution

$$
V = (nDt + C_1)^{3/n}, \quad n \neq 0, \quad C_1 = \text{const.}
\tag{3.13}
$$
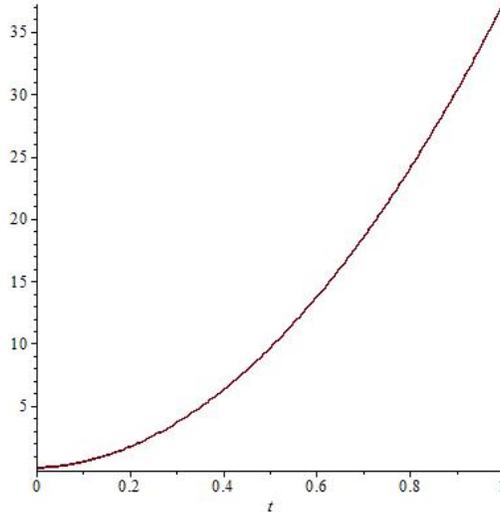
FIG. 1. Evolution of the Universe given by a BV cosmological model.

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Fig. [1] shows the evolution of the Universe. As one sees, it is an expanding one. The value of deceleration parameter is found to be

$$
q = n - 1,
\tag{3.14}
$$

which is a constant. The sign of q indicates whether the model inflates or not. The positive sign of q i.e. $(n > 1)$ correspond to "standard" decelerating model whereas the negative sign of q i.e. $0 \leq n < 1$ indicates inflation. It is remarkable to mention here that though the current observations of SNe Ia and CMBR favours accelerating models $(q < 0)$, but both do not altogether rule out the decelerating ones which are also consistent with these observations [26].

IV.   PHYSICAL ASPECTS OF DARK ENERGY MODEL

Let us now find the expressions for physical quantities.
Inserting (3.12) into (2.15) one finds the expression for expansion ϑ, Hubble parameter H:

$$\vartheta = 3H = \frac{3D}{nDt + C_1},$$

(4.1)
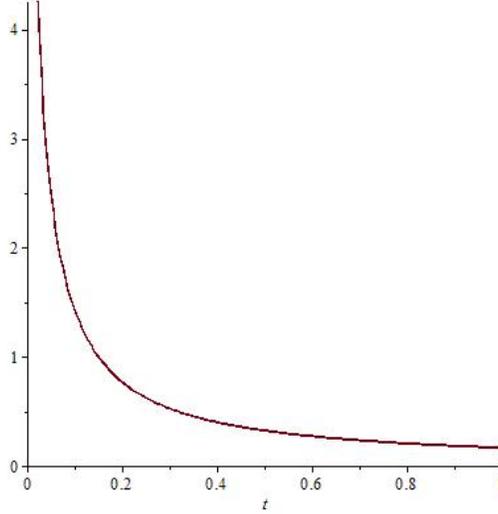
FIG. 2. Evolution of the Hubble parameter.

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Fig. [2] shows the evolution of the Hubble parameter. As one sees, it is a decreasing function of time.
The value of deceleration parameter is found to be

$$q = n - 1,$$

(4.2)

which is a constant. The sign of q indicates whether the model inflates or not. The positive sign of q i.e. $(n > 1)$ correspond to "standard" decelerating model whereas the negative sign of q i.e. $0 \le n < 1$ indicates inflation. It is remarkable to mention here that though the current observations of SNe Ia and CMBR favours accelerating models ($q < 0$), but both do not altogether rule out the decelerating ones which are also consistent with these observations [26].
The anisotropy parameter A_m has the expression

$$A_m = \frac{6}{N_1^2}.$$

(4.3)

The directional Hubble parameters are

$$H_1 = \left(\frac{1}{3} + \frac{1}{N_1}\right)\frac{3D}{nDt + C_1}, \quad H_2 = \left(\frac{1}{3} - \frac{1}{N_1}\right)\frac{3D}{nDt + C_1}, \quad H_3 = \frac{D}{nDt + C_1}.$$

(4.4)

From (2.3d) we find the expression for energy density For energy density in this case we have

$$\varepsilon = \frac{X_1}{(nDt + C_1)^2} - \frac{3m^2 C_1^2}{(nDt + C_1)^{2/n}},$$

(4.5)

where $X_1 = 9D^2(1/3 - 1/N_1^2)$. The EoS parameter in this case has the form

$$\omega = \frac{X_2/(nDt + C_1)^2 + m^2 C_1^2/(nDt + C_1)^{2/n}}{X_1/(nDt + C_1)^2 - 3m^2 C_1^2/(nDt + C_1)^{2/n}}, \qquad (4.6)$$

where $X_2 = X_1 - 2D^2(3 - n)$.
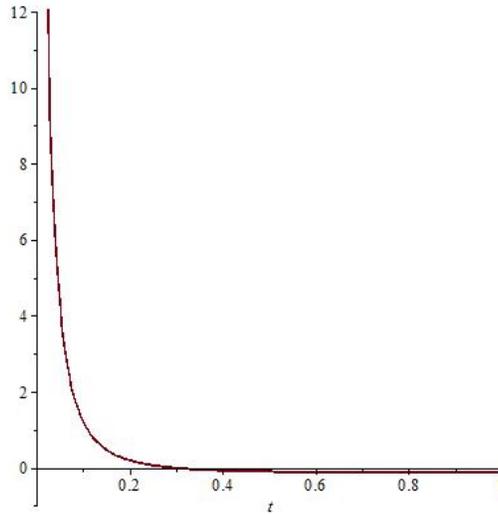
FIG. 3. Evolution of energy density.

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Fig. [3] shows the evolution of energy density. As one sees, it is a decreasing function of time and beginning some moment of time it may be negative as well.
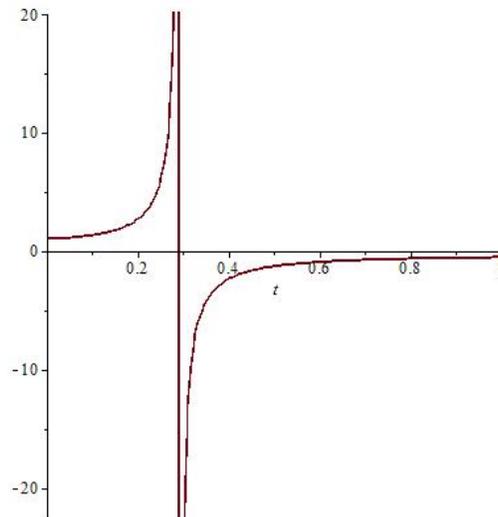
FIG. 4. Evolution of the EoS parameter.

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Fig. [4] shows the evolution of the EoS parameter. As one sees, it is a time varying function and changes its sign in the course of evolution.

From equation (4.6), it is observed that the equation of state parameter ω is time dependent, it can be function of redshift z or scale factor a as well. The redshift dependence of ω can be linear like

$$\omega(z) = \omega_0 + \omega' z, \tag{4.7}$$

with $\omega' = \frac{d\omega}{dz}\big|_{z=0}$ (see Refs. [9, 28] or nonlinear as [5, 14]

$$\omega(z) = \omega_0 + \frac{\omega_1 z}{1+z}. \tag{4.8}$$

So, as for as the scale factor dependence of ω is concern, the parametrization

$$\omega(a) = \omega_0 + \omega_a(1-a), \tag{4.9}$$

where ω_0 is the present value ($a = 1$) and ω_a is the measure of the time variation ω' is widely used in the literature [15].

Let us now compare the our results with the experimental results obtained in [7, 12, 13, 24]. It enable us to conclude that the limit of ω provided by equation (4.6) may accommodated with the acceptable range of EoS parameter. Also it is observed that at $t = t_c$, ω vanishes, where t_c is a critical time given by

$$t_c = \frac{1}{nD}\left[\left(\frac{X_2}{m^2 C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.10}$$

Thus, for this particular time, our model represents a dusty universe. We also note that the earlier real matter at $t \leq t_c$, where $\omega \geq 0$ later on at $t > t_c$, where $\omega < 0$ converted to the dark energy dominated phase of universe.

For the value of ω to be in consistent with observation [12], we have the following general condition

$$t_1 < t < t_2, \tag{4.11}$$

where

$$t_1 = \frac{1}{nD}\left[\left(\frac{X_2 + 1.67X_1}{-4.01m^2 C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.12}$$

and

$$t_1 = \frac{1}{nD}\left[\left(\frac{X_2 + 0.62X_1}{-0.86m^2 C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.13}$$

For this constrain, we obtain $-1.67 < \omega < -0.62$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [12].

For the value of ω to be in consistent with observation [24], we have the following general condition

$$t_3 < t < t_4, \tag{4.14}$$

where

$$t_1 = \frac{1}{nD}\left[\left(\frac{X_2 + 1.33X_1}{-2.99m^2 C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.15}$$

and

$$t_1 = \frac{1}{nD}\left[\left(\frac{X_2 + 0.79X_1}{-1.37m^2 C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.16}$$

For this constrain, we obtain $-1.33 < \omega < -0.79$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [24].

For the value of ω to be in consistent with observation [7, 13], we have the following general condition

$$t_5 < t < t_6, \tag{4.17}$$

where

$$t_1 = \frac{1}{nD}\left[\left(\frac{X_2 + 1.44X_1}{-3.32m^2C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.18}$$

and

$$t_1 = \frac{1}{nD}\left[\left(\frac{X_2 + 0.92X_1}{-1.76m^2C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.19}$$

For this constrain, we obtain $-1.44 < \omega < -0.92$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [7, 13].

We also observed that if

$$t_1 = \frac{1}{nD}\left[\left(\frac{X_2 + X_1}{-2m^2C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.20}$$

then for $t = t_0$ we have $\omega = -1$, i.e., we have universe with cosmological constant. If $t < t_0$ the we have $\omega > -1$ that corresponds to quintessence, while for $t > t_0$ we have $\omega > -1$, i.e., Universe with phantom matter [6].

From (4.5) we found that the energy density is a decreasing function of time and $\varepsilon \geq 0$ when

$$t \geq \frac{1}{nD}\left[\left(-\frac{X_1}{m^2C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.21}$$

In absence of any curvature, matter energy density Ω_m and dark energy density Ω_Λ are related by the equation

$$\Omega_m + \Omega_\Lambda = \frac{\varepsilon}{3H^2} + \frac{\Lambda}{3H^2} = 1. \tag{4.22}$$

Inserting (4.1) and (4.5) into (4.22) we find the cosmological constant as

$$\Lambda = \frac{3D^2 - X_1}{(nDt + C_1)^2} + \frac{3m^2C_1^2}{(nDt + C_1)^{2/n}}, \tag{4.23}$$

As we see, the cosmological function is a decreasing function of time and it is always positive when

$$t \geq \frac{1}{nD}\left[\left(\frac{X_1 - 3D^2}{3m^2C_1^2}\right)^{n/2(n-1)} - C_1\right]. \tag{4.24}$$

Fig. [5] shows the evolution of the cosmological constant. As one sees, it is a time varying function and decreases with time.

Recent cosmological observations suggest the existence of a positive cosmological constant Λ with the magnitude $\Lambda(G\hbar/c^3) \approx 10^{-123}$. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. Thus, the nature of Λ in our derived DE model is supported by recent observations.

For the stability of corresponding solutions, we should check that our models are physically acceptable. For this, the velocity of sound is less than that of light, i.e.,

$$0 \leq v_s = \frac{dp}{d\varepsilon} < 1. \tag{4.25}$$

In this case we find

$$v_s = \frac{dp}{d\varepsilon} = \frac{nX_2 + m^2C_1^2(nDt + C_1)^{2-2/n}}{nX_1 - 3m^2C_1^2(nDt + C_1)^{2-2/n}}. \tag{4.26}$$

Fig. [6] shows the behavior of v_s in time.

As one sees, there are regions, where the solution is stable. Fig. [6] shows that the solution becomes unstable during the transition from deceleration to acceleration phase of evolution. Choosing the problem parameters, such as n, D we can obtain the stable solutions before or after the transition.
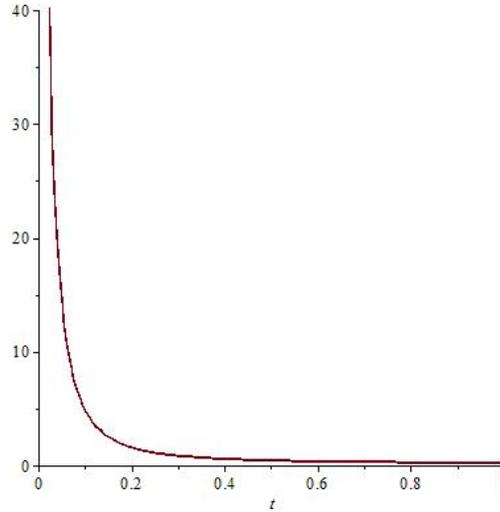
FIG. 5. Evolution of the EoS parameter.

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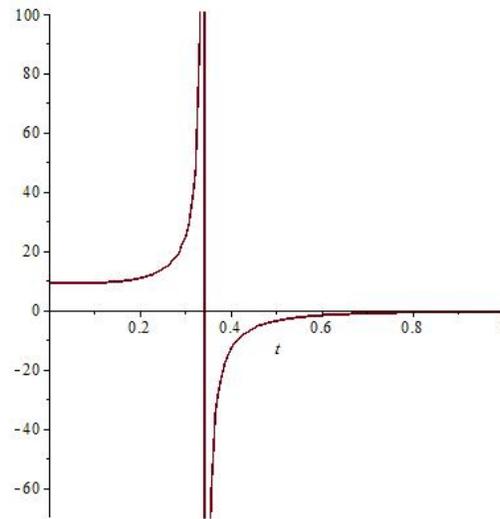
FIG. 6. Speed of sound with respect to cosmic time

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V.   CONCLUSION

In this report we have studied the evolution of the universe filled with dark energy within the scope of a Bianchi type-V model. Exact solutions to the field equations are obtained using the proportionality condition and variational law of Hubble parameter. It was found that the assumption of diagonal energy-momentum tensor together with the non-diagonal Einstein equation leads to some restriction on the energy momentum tensor, namely, $T_1^1 + T_2^2 = 2T_3^3$. The behavior of EoS parameter ω is thoroughly studied. It is found that the solution becomes stable as the Universe expands.

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