

# Black Hole Bound State Metamorphosis

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## Abstract

$\mathcal{N} = 4$  supersymmetric string theories contain negative discriminant states whose numbers are known precisely from microscopic counting formulæ. On the macroscopic side, these results can be reproduced by regarding these states as multi-centered black hole configurations provided we make certain identification of apparently distinct multi-centered black hole configurations according to a precise set of rules. In this paper we provide a physical explanation of such identifications, thereby establishing that multi-centered black hole configurations reproduce correctly the microscopic results for the number of negative discriminant states without any ad hoc assumption.

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## 1 Introduction

Matching of microscopic counting of BPS states to the entropy of supersymmetric black holes is an important problem. Exact microscopic counting of BPS states, including the dependence of the spectrum on the asymptotic moduli, has now been achieved for a wide class of states in  $\mathcal{N} = 8$  supersymmetric string theories [1–3] and a wide class of  $\mathcal{N} = 4$  supersymmetric string theories [4–18] in four dimensions. An important class of these microscopic states are the so called negative discriminant states – states carrying charges for which there are no classical supersymmetric single centered black holes but whose microscopic index is nevertheless non-zero. In particular such states are abundant in  $\mathcal{N} = 4$  supersymmetric string theories. It was shown in [19], following an earlier observation of [17], that all the known negative discriminant states in  $\mathcal{N} = 4$  supersymmetric string theories, which appear in the microscopic counting of states, can be accounted for precisely as 2-centered black hole configurations, with each center representing a small half-BPS black hole. This however required one crucial assumption: certain 2-centered configurations, whose indices can be computed and shown to

be the same, had to be treated as identical configurations. This identification was ad hoc, since the configurations which had to be identified appeared to be different configurations carrying the same total charge. Nevertheless [19] gave a precise set of rules for determining when a pair of configurations have to be identified. This phenomenon was called black hole metamorphosis. A similar phenomenon in the context of supersymmetric gauge theories had been discussed earlier in [20].

Our main goal in this paper will be to understand the physical origin of this phenomenon, and justify the ad hoc prescription of [19] for identifying certain apparently different configurations of black holes. What we shall show is that precisely for the class of two centered solutions for which the ad hoc identification rule is to be imposed, one of the black hole centers need to be replaced by a smooth gauge theory dyon to avoid certain singularities in the solution. The effect of this is that the range of moduli for which each solution exists is smaller than the one based on the naive analysis of a two centered black hole solution. Taking into account this effect, we find that at any given point of the moduli space the total index contributed by all the two centered configurations which exist at that point adds up to match precisely the microscopic result for the same index. Although we have carried out our analysis in the context of a specific theory – for heterotic string theory on  $T^6$  – and worked in a region of the moduli space where one of the two centers is light and can be regarded as a test particle in the background produced by the other center, we expect that our analysis captures the essential physics behind the phenomenon of black hole bound state metamorphosis for more general theories and in generic region of the moduli space.

## 2 Review of black hole metamorphosis

In this section we shall review the phenomenon of black hole bound state metamorphosis discussed in [19]. Although this phenomenon takes place in all  $\mathcal{N} = 4$  supersymmetric string theories, we shall consider in this paper the concrete example of heterotic string theory compactified on  $T^6$ . At a generic point in the moduli space this theory has 28 U(1) gauge fields and hence a BPS state is characterized by a 28 dimensional electric charge vector  $Q$  and a 28 dimensional magnetic charge vector  $P$ . We shall denote the combined charge vector as  $(Q, P)$ . We can associate with these vectors T-duality invariant bilinears  $Q^2$ ,  $P^2$  and  $Q \cdot P$ . We consider quarter BPS states carrying charges  $(\widehat{Q}, \widehat{P})$  satisfying

$$(\widehat{Q}^2 \widehat{P}^2 - (\widehat{Q} \cdot \widehat{P})^2) < 0, \quad \text{and} \quad \text{gcd}\{\widehat{Q}_i \widehat{P}_j - \widehat{Q}_j \widehat{P}_i, \quad 1 \leq i, j \leq 28\} = 1. \quad (2.1)$$

In this case there are no single centered black holes carrying these charges and the only two centered configurations which can contribute to the index carry charges of the form:

$$(aQ, cQ) \quad \text{and} \quad (bP, dP), \quad (2.2)$$

for some vectors  $Q$  and  $P$  and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ , carrying total charge  $(aQ+bP, cQ+dP) = (\widehat{Q}, \widehat{P})$ . This two centered configuration exists in a certain region of the moduli space of the theory determined by the rules given in [19]. Outside this region the configuration ceases to exist and hence does not contribute to the index. The contribution to the index from this configuration when it exists is given by

$$(-1)^{Q \cdot P + 1} |Q \cdot P| d_h(Q^2/2) d_h(P^2/2), \quad (2.3)$$

where

$$\sum_n d_h(n) q^n = q^{-1} \prod_{k=1}^{\infty} (1 - q^k)^{-24}. \quad (2.4)$$

Physically  $d_h(n)$  denotes the index of half BPS states.

The phenomenon of metamorphosis takes place when either  $P^2$  or  $Q^2$  (or both) take the value  $-2$ . Let us suppose  $P^2 = -2$ . In that case the configuration

$$(a'Q', c'Q') \quad \text{and} \quad (b'P', d'P'), \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \equiv \begin{pmatrix} a & b - au \\ c & d - cu \end{pmatrix}, \quad Q' \equiv Q + uP, \quad P' \equiv P, \quad u \equiv Q \cdot P \quad (2.5)$$

has the same total charge, satisfies

$$Q'^2 = Q^2, \quad P'^2 = P^2, \quad Q' \cdot P' = -Q \cdot P, \quad (2.6)$$

and hence, according to (2.3) gives the same contribution to the index. Now suppose that the configuration (2.2) exists in the region  $R_1$  in the moduli space and the configuration (2.5) exists in the region  $R_2$ . It turns out that  $R_1 \cup R_2$  spans the whole moduli space of the theory. Thus naively one would expect that in the region  $R = R_1 \cap R_2$  the total contribution to the index from these two configurations will be given by twice of (2.3) whereas outside this region the contribution to the index will be given by (2.3). *However in order to match the microscopic result we have to assume that in the region  $R$  the contribution to the index is given by (2.3) while outside this region there is no contribution to the index from these configurations.*

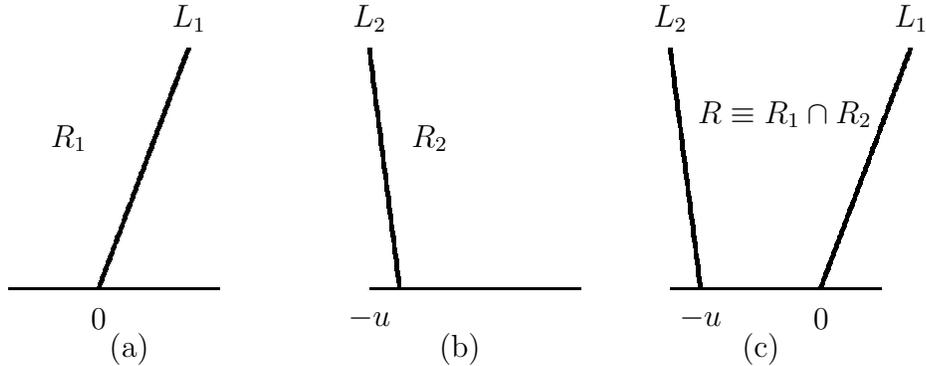


Figure 1: Figure illustrating the walls of marginal stability and the region of existence of the configurations described in (2.7) and (2.8). In Fig. (a) the thick line  $L_1$  labels the wall of marginal stability for the configuration (2.7) which exists in the region  $R_1$  to the left of  $L_1$  in the upper half plane. In Fig. (b) the thick line  $L_2$  labels the wall of marginal stability for the configuration (2.8) which exists in the region  $R_2$  to the right of  $L_2$  in the upper half plane. Fig. (c) labels the region  $R \equiv R_1 \cap R_2$ . Thus naively we expect both configurations to exist in the region  $R$  and one of the two configurations to exist in the region outside  $R$ . However microscopic counting requires that only one of the two configurations exist in the region  $R$  and none exist outside this region. In drawing these figures we have implicitly taken  $u$  to be positive. If  $u$  is negative then each figure has to be reflected about the vertical axis passing through the origin.

The case where  $Q^2 = -2$  is related to the above by a duality transformation and need not be discussed separately. In fact with the help of an S-duality transformation by the matrix  $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$  we can map the configurations (2.2) and (2.5) to

$$(Q, 0) \quad \text{and} \quad (0, P), \quad P^2 = -2, \quad (2.7)$$

and

$$(Q + uP, 0) \quad \text{and} \quad (-uP, P), \quad u \equiv Q \cdot P, \quad (2.8)$$

with each configuration carrying the same index as (2.3). Thus we shall focus on this configuration from now on. In this case Fig. 1 shows the regions  $R_1$ ,  $R_2$  and  $R$  in the upper half  $\tau$ -plane [16] – where  $\tau = \tau_1 + i\tau_2$  denotes the asymptotic values of the axion-dilaton modulus of the heterotic string theory on  $T^6$ – for fixed asymptotic values of the other fields. The boundaries of  $R_1$ ,  $R_2$  marked by the thick lines  $L_1$  and  $L_2$  correspond to walls of marginal stability beyond which the configurations (2.7) and (2.8) cease to exist. The precise slope of these straight lines depend on the details of the charges and the asymptotic values of the other moduli, and will be given in eqs.(4.24) and (4.32) respectively.

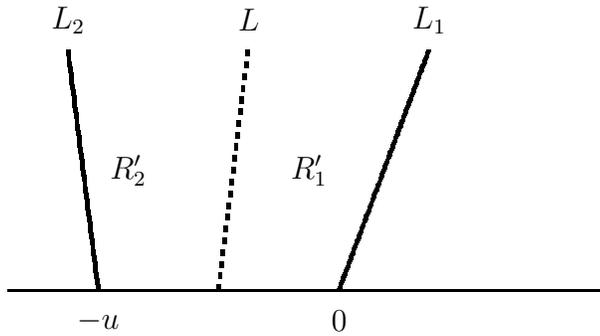


Figure 2: The pictorial description of black hole metamorphosis.

The phenomenon of black hole metamorphosis suggests the existence of a hypothetical line  $L$ , shown in Fig. 2, such that the configuration (2.7) exists only in the region  $R'_1$  to the right of  $L$  and left of  $L_1$  and the configuration (2.8) exists only in the region  $R'_2$  to the left of  $L$  and the right of  $L_2$ . In that case it would explain why the index is given by (2.3) in the region  $R'_1 \cup R'_2 = R$  and vanishes outside this region. Our goal will be to understand the physical origin of  $L$ .

### 3 Review of multi-black hole solutions in $\mathcal{N} = 2$ supergravity

Although heterotic string theory on  $T^6$  describes an  $\mathcal{N} = 4$  supersymmetric string theory, the multi-black hole solutions are best understood in the language of  $\mathcal{N} = 2$  supergravity. For this reason in this section we shall review multi-black hole solutions in  $\mathcal{N} = 2$  supergravity. The bosonic fields of an  $\mathcal{N} = 2$  supergravity coupled to  $n_v$  vector multiplet fields are the metric  $g_{\mu\nu}$ ,  $n_v + 1$  complex scalars  $X^I$ , and  $n_v + 1$  gauge fields  $\mathcal{A}_\mu^I$  with  $0 \leq I \leq n_v$ . The theory has a complex gauge invariance under which all the  $X^I$ 's scale by an arbitrary complex function  $\Lambda(x)$ , the metric scales by  $|\Lambda|^{-2}$  and the gauge fields remain invariant. The action of the theory is completely fixed by the prepotential  $F$  which is a meromorphic, homogeneous function of the  $X^I$ 's of degree 2. If  $(q, p)$  denote the electric and magnetic charge vectors carried by a state with  $q$  and  $p$  being  $n_v + 1$  dimensional vectors, then we define

$$F_I \equiv \partial F / \partial X^I, \quad e^{-K} \equiv i(\bar{X}^I F_I - X^I \bar{F}_I), \quad Z(q, p) \equiv (q_I X^I - p^I F_I) e^{K/2}. \quad (3.1)$$

The gauge fields are normalized so that the action of a test particle carrying electric charges  $\hat{q}_I$  and magnetic charges  $\hat{p}^I$  takes the form

$$\frac{1}{2} \int (\hat{q}_I \mathcal{A}_\mu^I - \hat{p}^I \mathcal{A}_{I\mu}) dx^\mu \quad (3.2)$$

where  $\mathcal{A}_\mu^I$  denotes the usual gauge potential and  $\mathcal{A}_{I\mu}$  denotes the dual magnetic potential.

A general supersymmetric multi-centered black hole solution in such a theory was constructed in [21, 22]. To describe the solution we introduce the functions:

$$H^I = \sum_i \frac{p_{(i)}^I}{|\vec{r} - \vec{r}_i|} - 2 \operatorname{Im} [e^{-i\alpha_\infty} X^I e^{K/2}]_\infty, \quad H_I = \sum_i \frac{q_{(i)I}}{|\vec{r} - \vec{r}_i|} - 2 \operatorname{Im} [e^{-i\alpha_\infty} F_I e^{K/2}]_\infty, \quad (3.3)$$

where  $\vec{r}_i$  are the locations of the centers in the three dimensional space,  $(q_{(i)}, p_{(i)})$  denote the electric and magnetic charges carried by the  $i$ -th center, the subscript  $\infty$  denotes the asymptotic values of the various fields and

$$\alpha_\infty = \operatorname{Arg} \left[ Z \left( \sum_i q_{(i)}, \sum_i p_{(i)} \right) \right]_\infty. \quad (3.4)$$

Now let

$$S_{BH}(\{q_I\}, \{p^I\}) = \pi \Sigma(\{q_I\}, \{p^I\}), \quad (3.5)$$

be the entropy of a single centered black hole solution in this theory with charge  $(q, p)$ . There is a standard algorithm for computing the function  $\Sigma$  from the knowledge of the function  $F$  – it is given by the extremum of  $|Z(q, p)|^2$  with respect to the scalar moduli fields. We now define

$$\chi^K(\{q_I\}, \{p^I\}) \equiv -\frac{\partial \Sigma}{\partial q_K}, \quad \chi_K(\{q_I\}, \{p^I\}) \equiv \frac{\partial \Sigma}{\partial p^K}, \quad (3.6)$$

$$g^K(\{q_I\}, \{p^I\}) = \chi^K(\{q_I\}, \{p^I\}) + ip^K, \quad g_K(\{q_I\}, \{p^I\}) = \chi_K(\{q_I\}, \{p^I\}) + iq_K. \quad (3.7)$$

Then the solution for the scalar fields, metric and the gauge fields is given by

$$\frac{X^K}{X^0} = \frac{g^K(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})}{g^0(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})}, \quad \frac{F_K}{X^0} = \frac{g_K(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})}{g^0(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})}, \quad (3.8)$$

$$ds^2 = e^{2V} (dt + \vec{\omega} \cdot d\vec{x})^2 + e^{-2V} dx^i dx^i, \quad (3.9)$$

$$e^{-2V} \equiv \Sigma(\{H_I(\vec{r})\}, \{H^I(\vec{r})\}), \quad (3.10)$$

$$\begin{aligned}
\mathcal{A}_\mu^K dx^\mu &= -\Sigma(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})^{-1} \chi^K(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})(dt + \vec{\omega} \cdot d\vec{x}) - \sum_i p_{(i)}^K \cos \theta_{(i)} d\phi_{(i)}, \\
\mathcal{A}_{K\mu} dx^\mu &= -\Sigma(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})^{-1} \chi_K(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})(dt + \vec{\omega} \cdot d\vec{x}) - \sum_i q_{(i)K} \cos \theta_{(i)} d\phi_{(i)}
\end{aligned} \tag{3.11}$$

where  $(\theta_{(i)}, \phi_{(i)})$  denote the polar and azimuthal angles of the spherical polar coordinate system with origin at  $\vec{r}_i$ . The general solution for  $\vec{\omega}$  exists but we shall not need it. For single centered solution  $\vec{\omega}$  vanishes. Finally consistency demands that the locations  $\vec{r}_i$  be subject to the constraint:

$$\sum_{\substack{j=1 \\ j \neq i}}^n \frac{q_{(i)I} p_{(j)}^I - q_{(j)I} p_{(i)}^I}{|\vec{r}_i - \vec{r}_j|} = 2 \operatorname{Im}(e^{-i\alpha_\infty} Z_i), \quad Z_i \equiv Z(q_{(i)}, p_{(i)})|_\infty \tag{3.12}$$

One clarification is necessary here. The combinations  $X^I/X^0$ ,  $F_I/X^0$  and the gauge fields are invariant under the complex gauge transformation generated by  $\Lambda(x)$  and hence it is not necessary to specify in which gauge we have given the solutions. However since the metric is not invariant under this transformation we need to specify the gauge in which the metric is given. (3.9) is given in the choice of gauge in which the Einstein-Hilbert term takes the form [22]

$$\frac{1}{16\pi} \int d^4x \sqrt{-\det g} R. \tag{3.13}$$

For a 2-centered solution carrying charges  $(q_{(1)}, p_{(1)})$  at  $\vec{r}_1$  and  $(q_{(2)}, p_{(2)})$  at  $\vec{r}_2$ , (3.12) gives

$$|\vec{r}_1 - \vec{r}_2| = \frac{q_{(2)I} p_{(1)}^I - q_{(1)I} p_{(2)}^I}{2 \operatorname{Im}(e^{-i\alpha_\infty} Z_2)}, \quad e^{i\alpha_\infty} = \frac{Z_1 + Z_2}{|Z_1 + Z_2|}. \tag{3.14}$$

When  $|Z(q_{(2)}, p_{(2)})|$  is small and we can ignore the background field produced by the second center in most of the space, then an independent way of arriving at the result (3.14) is as follows. Let us consider the background fields produced by a single centered solution carrying charges  $(q_{(1)}, p_{(1)})$  placed at the origin. If we now place a test particle carrying charge  $(\hat{q}, \hat{p})$  in this background then the action of this test particle takes the form

$$S_t = - \int d\tau |Z(\{\hat{q}_I\}, \{\hat{p}^I\})| + \frac{1}{2} \int (\hat{q}_I \mathcal{A}_\mu^I - \hat{p}^I \mathcal{A}_{I\mu}) dx^\mu \tag{3.15}$$

where  $\tau$  is the proper time, and  $x^\mu$  denote the coordinates of the test particle. If the test particle is at rest then we have  $d\tau = e^V dt$  and hence

$$S_t = \int dt \left[ -e^V |Z(\{\hat{q}_I\}, \{\hat{p}^I\})| + \frac{1}{2} (\hat{q}_I \mathcal{A}_0^I - \hat{p}^I \mathcal{A}_{I0}) \right]. \tag{3.16}$$

The equilibrium position of the test particle will be at the extremum of the integrand with respect to the spatial coordinates  $x^1, x^2, x^3$ . It can be shown that this gives us back (3.14) with  $(q_{(2)}, p_{(2)})$  replaced by  $(\hat{q}, \hat{p})$  if  $|Z(q_{(2)}, p_{(2)})|$  is small so that we can treat the second center as a test particle ignoring its backreaction on the geometry [21].

## 4 S-T-U model

In this section we shall analyze a class of 2-centered black hole solutions in heterotic string theory on  $T^6$  and propose a mechanism for black hole metamorphosis. Our analysis will proceed in several steps. First we shall describe a truncation of heterotic string theory on  $T^6$  which can be mapped to an  $\mathcal{N} = 2$  supergravity theory, known as the S-T-U model. We shall then describe the S-T-U model and the maps between the fields in the two descriptions. We then consider a two centered configuration in this theory with one center carrying charge  $(0, P)$  with  $P^2 = -2$ , and take a limit where the other center carrying charge  $(Q, 0)$  becomes light and can be regarded as a probe. The technique reviewed in §3 then enables us to easily construct the background field associated with the heavy center and find the equilibrium position of the light center. We then analyze the solution carefully to find the region of the moduli space where it exists. Although naively it exists in the region  $R_1$  to the left of the line  $L_1$  in Fig. 2, we suggest a mechanism by which the region of existence gets truncated to  $R'_1$  displayed in Fig. 2. We repeat the analysis for another configuration related to the first by the transformation (2.8) and show that this exists in the region  $R'_2$  as displayed in Fig. 2. This analysis also allows us to determine the precise location of the line  $L$  in Fig. 2.

### 4.1 Truncation of heterotic string theory on $T^6$

We shall now describe the truncation of heterotic string theory on  $T^6$  that can be mapped to an  $\mathcal{N} = 2$  supergravity theory. For this we take  $T^6$  in the form of the product  $T^4 \times T^2$  and ignore all excitations of the components of the metric and 2-form fields with one or both legs along  $T^4$  and also all excitations of the ten dimensional gauge fields. This truncated theory will have only four gauge fields corresponding to 4- $\mu$  and 5- $\mu$  components of the metric and the 2-form gauge fields, with  $x^4$  and  $x^5$  denoting the coordinates along  $T^2$  and  $x^\mu$  with  $0 \leq \mu \leq 3$  denoting the coordinates along the 3+1 dimensional non-compact space-time. The other relevant fields are the canonical metric  $g_{\mu\nu}$ , the axion dilaton modulus  $S = S_1 + iS_2$ , the complex structure modulus  $U = U_1 + iU_2$  of  $T^2$  and the complexified Kahler modulus  $T = T_1 + iT_2$  of  $T^2$ . The

four components  $(Q_1, Q_2, Q_3, Q_4)$  of the electric charge vector  $Q$  correspond respectively to the number of units of momentum along  $x^5$  and  $x^4$  respectively and winding numbers along  $x^5$  and  $x^4$  respectively. On the other hand the components  $(P_1, P_2, P_3, P_4)$  of the magnetic charge  $P$  denote respectively the heterotic five-brane winding numbers along  $T^4 \times x^4$ -circle and  $T^4 \times x^5$ -circle respectively and Kaluza-Klein monopole charges associated with  $x^5$  and  $x^4$  directions respectively. The bilinears  $Q^2, P^2, Q \cdot P$  are given by

$$Q^2 = 2(Q_1Q_3 + Q_2Q_4), \quad P^2 = 2(P_1P_3 + P_2P_4), \quad Q \cdot P = Q_1P_3 + Q_2P_4 + Q_3P_1 + Q_4P_2. \quad (4.1)$$

Finally the entropy of a black hole carrying (electric, magnetic) charges  $(Q, P)$  is given by

$$S_{BH} = \pi\sqrt{\Sigma}, \quad \Sigma = Q^2P^2 - (Q \cdot P)^2. \quad (4.2)$$

## 4.2 $\mathcal{N} = 2$ description

This truncated theory can be mapped to an  $\mathcal{N} = 2$  supergravity theory coupled to three vector multiplets, with prepotential

$$F = -\frac{X^1X^2X^3}{X^0}. \quad (4.3)$$

The scalar fields  $S, T$  and  $U$  introduced in §4.1 are given by

$$S = \frac{X^1}{X^0}, \quad T = \frac{X^2}{X^0}, \quad U = \frac{X^3}{X^0}. \quad (4.4)$$

The relations between the gauge fields in the two descriptions can be described by giving the relations between the charges  $\{Q_i, P_i\}$  given above with the charges  $\{q_I, p^I\}$  in the  $\mathcal{N} = 2$  supergravity description. This is as follows (see *e.g.* [23] for a review)

$$Q \equiv (Q_1, Q_2, Q_3, Q_4) = (q_0, q_3, -p^1, q_2), \quad P \equiv (P_1, P_2, P_3, P_4) = (q_1, p^2, p^0, p^3). \quad (4.5)$$

Eq.(4.2) now gives

$$\Sigma(\{q_I\}, \{p^I\}) = [4(q_2q_3 - q_0p^1)(p^0q_1 + p^2p^3) - (q_0p^0 - q_1p^1 + q_2p^2 + q_3p^3)^2]^{1/2}. \quad (4.6)$$

We shall denote the asymptotic values of the various moduli fields as

$$S|_\infty = \zeta \equiv \zeta_1 + i\zeta_2, \quad T|_\infty = \rho \equiv \rho_1 + i\rho_2, \quad U|_\infty = \sigma \equiv \sigma_1 + i\sigma_2. \quad (4.7)$$

As we shall see in (4.19),  $\zeta$  is related to the modulus  $\tau$  of §2 via the relation  $\zeta = -\bar{\tau}$ . We also define

$$x^0 \equiv X_\infty^0. \quad (4.8)$$

From (3.1), (4.3), (4.7) and (4.8) it follows that

$$e^{-K} = 8X^0 \bar{X}^0 S_2 T_2 U_2, \quad e^{-K}|_{\infty} = 8x^0 \bar{x}^0 \zeta_2 \sigma_2 \rho_2. \quad (4.9)$$

### 4.3 The two centered solution

In the asymptotic background described above we construct a two centered solution with the first center carrying charge  $(0, P)$  and the second center carrying charge  $(Q, 0)$ , with

$$Q = (a, b, c, d), \quad P = (0, -1, 0, 1). \quad (4.10)$$

This gives, from (4.1)

$$Q^2 = 2ac + 2bd, \quad P^2 = -2, \quad u \equiv Q \cdot P = b - d. \quad (4.11)$$

We shall for definiteness take  $(b - d) > 0$  so that  $u > 0$ . Since  $P^2 = -2$  this configuration should display the phenomenon of black hole bound state metamorphosis. In particular there must exist a line  $L$  in the  $\tau = -\bar{\zeta}$  plane such that the bound state ceases to exist to the left of this line. Our goal will be understand the physical origin of this hypothetical line  $L$ .

Now using (4.5) we see that in the language of  $\mathcal{N} = 2$  supergravity the two centers carry charges  $(q_{(1)}, p_{(1)})$  and  $(q_{(2)}, p_{(2)})$  where

$$p_{(1)} = (0, 0, -1, 1), \quad q_{(1)} = (0, 0, 0, 0), \quad p_{(2)} = (0, -c, 0, 0), \quad q_{(2)} = (a, 0, d, b). \quad (4.12)$$

We define

$$\begin{aligned} Z_1 &\equiv Z(q_{(1)}, p_{(1)})|_{\infty} = [e^{K/2}(F_2 - F_3)]_{\infty} = \sqrt{\frac{x^0}{\bar{x}^0}} \frac{1}{\sqrt{8\zeta_2 \rho_2 \sigma_2}} \zeta(\rho - \sigma), \\ Z_2 &\equiv Z(q_{(2)}, p_{(2)})|_{\infty} = [e^{K/2}(aX^0 + dX^2 + bX^3 + cF_1)]_{\infty} \\ &= \sqrt{\frac{x^0}{\bar{x}^0}} \frac{1}{\sqrt{8\zeta_2 \rho_2 \sigma_2}} (a + d\rho + b\sigma - c\rho\sigma). \end{aligned} \quad (4.13)$$

To simplify the analysis we shall work in the limit where  $\zeta_2$  is large. In this limit  $|Z_2|$  given in (4.13) is small showing that the corresponding state is light. Hence we can ignore its effect on the background field and treat this center as a probe. In this limit the background geometry approaches that of a single centered black hole with charge  $(q_{(1)}, p_{(1)})$  placed at  $\vec{r}_1$ , and  $\alpha_{\infty}$  defined in (3.4) and the functions  $H^I$  and  $H_I$  introduced in (3.3) take the form

$$e^{i\alpha_{\infty}} = \frac{Z_1 + Z_2}{|Z_1 + Z_2|} \simeq \frac{Z_1}{|Z_1|} = \sqrt{\frac{x^0}{\bar{x}^0}} \frac{\zeta}{|\zeta|} \frac{\rho - \sigma}{|\rho - \sigma|}, \quad (4.14)$$

$$(H^0, H^1, H^2, H^3) \simeq \frac{1}{|\vec{r} - \vec{r}_1|} (0, 0, -1, 1) - \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \text{Im} \left\{ \frac{|\zeta|}{\zeta} \frac{|\rho - \sigma|}{\rho - \sigma} (1, \zeta, \rho, \sigma) \right\}, \quad (4.15)$$

and

$$(H_0, H_1, H_2, H_3) \simeq \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \text{Im} \left\{ \frac{|\zeta|}{\zeta} \frac{|\sigma - \rho|}{\rho - \sigma} (-\zeta\rho\sigma, \rho\sigma, \zeta\sigma, \zeta\rho) \right\}. \quad (4.16)$$

From this we can construct the solution for the metric, scalars and gauge fields using the prescription reviewed in §3. The separation between the two centers is given, according to (3.14), by

$$|\vec{r}_1 - \vec{r}_2| = \frac{b - d}{2} \frac{\sqrt{8\zeta_2\sigma_2\rho_2}}{|\zeta||\sigma - \rho|} \frac{1}{\text{Im} [(a + d\rho + b\sigma - c\rho\sigma)/(\zeta(\rho - \sigma))]} . \quad (4.17)$$

Before we go on we must mention two subtle points in the relation between the  $\mathcal{N} = 4$  and  $\mathcal{N} = 2$  theory that will be important for our analysis. According to (4.13), the total mass of the system is given by

$$|Z_1 + Z_2| = \frac{1}{\sqrt{8\zeta_2\rho_2\sigma_2}} \sqrt{|A|^2 + |B|^2|\zeta|^2 + 2\zeta_1 \text{Re}(AB^*) + 2\zeta_2 \text{Im}(AB^*)},$$

$$A \equiv a + d\rho + b\sigma - c\rho\sigma, \quad B \equiv (\rho - \sigma). \quad (4.18)$$

Now consider a state carrying total charge  $(P, P)$ . The BPS mass of this state will be given by setting  $a = c = 0$  and  $b = -1, d = 1$  in (4.18) and its dependence on the axion dilaton modulus  $\zeta$  will be proportional to  $\sqrt{|1 + \zeta|^2}/\sqrt{\zeta_2}$ . On the other hand in the convention of [16, 19] which we used in presenting the results in §2, the dependence of the BPS mass of a particle of charge  $(P, P)$  on the axion dilaton modulus is proportional to  $\sqrt{|1 - \tau|^2}/\sqrt{\tau_2}$ . This shows that  $\zeta$  and  $\tau$  are related by

$$\zeta = -\bar{\tau}. \quad (4.19)$$

To discuss the second subtlety, let us return to the general formula (4.18). The BPS mass formula in the  $\mathcal{N} = 4$  supersymmetric theories (derived in [24, 25] and used *e.g.* in [16] for the analysis of the walls of marginal stability) is given by the same formula as (4.18) (after the identification (4.19)) except that the coefficient of  $\zeta_2 = \tau_2$  under the square root is given by  $2|\text{Im}(AB^*)|$ . Thus the two formulæ agree when  $\text{Im}(AB^*) > 0$ , i.e.

$$(\sigma_2 - \rho_2)(a + d\rho_1 + b\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2) + (\rho_1 - \sigma_1)(d\rho_2 + b\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)) > 0. \quad (4.20)$$

From now on we shall assume that this condition holds.

## 4.4 The region of existence of the solution

From (4.17) we can identify the wall of marginal stability as the curve in the  $\zeta$  plane on which the right hand side of (4.17) diverges. This gives

$$\frac{\zeta_1}{\zeta_2} = \frac{N}{D}, \quad \text{i.e.} \quad \frac{\tau_1}{\tau_2} = -\frac{N}{D}, \quad (4.21)$$

where

$$\begin{aligned} N &= -(\sigma_2 - \rho_2)(d\rho_2 + b\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)) + (\rho_1 - \sigma_1)(a + d\rho_1 + b\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2), \\ D &= (\sigma_2 - \rho_2)(a + d\rho_1 + b\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2) + (\rho_1 - \sigma_1)(d\rho_2 + b\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)). \end{aligned} \quad (4.22)$$

(4.21) marks the location of the line  $L_1$  in Figs. 1 and 2. In particular the solution exists when the right hand side of (4.17) is positive, i.e. for

$$\begin{aligned} \zeta_1 &> \frac{N}{D}\zeta_2 \quad \text{for } (b-d)D > 0, \\ &< \frac{N}{D}\zeta_2 \quad \text{for } (b-d)D < 0. \end{aligned} \quad (4.23)$$

Since according to (4.20) we have  $D > 0$ , and we have assumed that  $b - d > 0$ , this condition translates to

$$\zeta_1 > \frac{N}{D}\zeta_2 \quad \text{i.e.} \quad \tau_1 < -\frac{N}{D}\tau_2. \quad (4.24)$$

Naively one may expect that (4.24) is the only condition on  $\tau$  for the existence of the solution, since as long as (4.24) is satisfied,  $|\vec{r}_1 - \vec{r}_2|$  given in (4.17) remains positive. However upon closer examination one discovers a peculiar property of the solution that can be attributed to the special charge vector carried by the first center. If we take a test particle of charge  $(P, 0)$  with  $P = (0, -1, 0, 1)$  as in (4.10), it maps to  $q = (0, 0, 1, -1)$ ,  $p = (0, 0, 0, 0)$  in the  $\mathcal{N} = 2$  supergravity variables, and its mass at a point  $\vec{r}$  is given by

$$\frac{1}{8\sqrt{S_2(\vec{r})T_2(\vec{r})U_2(\vec{r})}}|T(\vec{r}) - U(\vec{r})|. \quad (4.25)$$

Thus it vanishes when  $T(\vec{r}) = U(\vec{r})$ . Using eqs.(3.6)-(3.8) and (4.6) we see that this requires  $H_2 = H_3$  and  $H^2 = H^3$ . Now from (4.16) we see that the first condition is satisfied automatically, while eq.(4.15) tells us that we have  $H^2 = H^3$  when

$$|\vec{r} - \vec{r}_1| = r_e, \quad r_e \equiv \sqrt{8\zeta_2\sigma_2\rho_2} \frac{|\zeta|}{\zeta_2|\rho - \sigma|}. \quad (4.26)$$

This describes a spherical shell of radius  $r_e$  around  $\vec{r}_1$  on which an electrically charged test particle carrying charge  $(P, 0)$  becomes massless. Physically on this shell the radius of the  $x^4$  direction reaches the self-dual point and hence we have massless non-abelian gauge fields. This in turn shows that at this point the original solution describing the background field produced by the charge  $(0, P)$  breaks down and we should not trust the solution for values of  $\vec{r}$  for which  $|\vec{r} - \vec{r}_1|$  is less than  $r_e$  defined in (4.26). This has been named the enhancon mechanism in [26]. Indeed, if we ignore this effect and continue to trust the solution for  $|\vec{r} - \vec{r}_1| < r_e$ , then at some point  $\Sigma(\{H_I\}, \{H^I\})$  computed from (4.6), (4.15) and (4.16) vanishes and the solution becomes singular [26]. We shall call  $r_e$  the enhancon radius. Thus a two centered solution, obtained by placing in the above background a test charge  $(Q, 0)$  at  $\vec{r}_2$  is sensible only when we have

$$|\vec{r}_1 - \vec{r}_2| > r_e. \quad (4.27)$$

Using (4.17), (4.26) and the positivity of  $D$  and  $b - d$ , this translates to

$$\zeta_1 < \frac{b-d}{2D} \zeta_2 |\rho - \sigma|^2 + \frac{N}{D} \zeta_2 \quad \text{i.e.} \quad \tau_1 > -\frac{b-d}{2D} \tau_2 |\rho - \sigma|^2 - \frac{N}{D} \tau_2. \quad (4.28)$$

As we shall see in §5, the correct description of the solution involves replacing it by a gravitationally dressed smooth BPS dyon obtained by boosting the Harvey-Liu monopole solution [27, 28] in an internal compact direction. As a result the solution begins to differ from that given in §4.3 even for  $|\vec{r} - \vec{r}_1| > r_e$ . However for now we shall take the above bound on  $\zeta_1$  seriously and examine its consequences. In this case (4.28) gives us the location of the left boundary  $L$  of the region  $R'_1$  in Fig. 2, with the right boundary  $L_1$  of  $R'_1$  being given by the wall of marginal stability (4.24). In §5 we shall see that this in fact is the exact result for the allowed range of  $\zeta_1$  in the large  $\zeta_2$  limit.

## 4.5 The second two centered solution

Next consider the two centered configuration with charges  $(-uP, P)$  and  $(Q + uP, 0)$  where  $u = Q \cdot P = (b - d)$ . Again one can argue that in the limit of large  $\zeta_2$  the second center carrying only electric charge  $(Q + uP, 0)$  is light and hence can be treated as a test particle. Furthermore, for the first center the contribution to the background field from the electric component proportional to  $uP$  will be small and hence can be dropped. Thus the problem effectively reduces to studying the test charge  $(Q + uP, 0)$  in the background produced by the charge  $(0, P)$ . Since according to (4.10), (4.11),  $Q + uP$  differs from  $Q$  just by the exchange

of the quantum numbers  $b$  and  $d$ , we can derive the various results for this system simply by exchanging  $b$  and  $d$  in the earlier results. In particular for this system the separation between the two centers is given by

$$|\vec{r}_1 - \vec{r}_2| = \frac{d-b}{2} \frac{\sqrt{8\zeta_2\sigma_2\rho_2}}{|\zeta||\sigma-\rho|} \frac{1}{\text{Im}[(a+b\rho+d\sigma-c\rho\sigma)/(\zeta(\rho-\sigma))]} . \quad (4.29)$$

The wall of marginal stability, where  $|\vec{r}_1 - \vec{r}_2|$  diverges, is at

$$\zeta_1 = \frac{N'}{D'}\zeta_2 \quad \text{i.e.} \quad \tau_1 = -\frac{N'}{D'}\tau_2 , \quad (4.30)$$

where

$$\begin{aligned} N' &= -(\sigma_2 - \rho_2)(b\rho_2 + d\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)) + (\rho_1 - \sigma_1)(a + b\rho_1 + d\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2) , \\ D' &= (\sigma_2 - \rho_2)(a + b\rho_1 + d\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2) + (\rho_1 - \sigma_1)(b\rho_2 + d\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)) \\ &= D . \end{aligned} \quad (4.31)$$

(4.30) marks the location of the line  $L_2$  in Figs.1 and 2. The solution exists for

$$\zeta_1 < \frac{N'}{D'}\zeta_2 \quad \text{i.e.} \quad \tau_1 > -\frac{N'}{D'}\tau_2 . \quad (4.32)$$

The enhancon radius remains at the same place as before. The condition that the location of the second center lies outside the enhancon radius can be translated to

$$\zeta_1 > \frac{d-b}{2D}\zeta_2|\rho-\sigma|^2 + \frac{N'}{D}\zeta_2 \quad \text{i.e.} \quad \tau_1 < \frac{b-d}{2D}\tau_2|\rho-\sigma|^2 - \frac{N'}{D}\tau_2 . \quad (4.33)$$

As before we shall take this to be our estimate for the right boundary of the region  $R'_2$  in Fig. 2, with the left boundary  $L_2$  of  $R'_2$  being given by the constraint (4.32).

We now note that

$$\frac{b-d}{2D}|\rho-\sigma|^2 - \frac{N'}{D} = \frac{d-b}{2D}|\rho-\sigma|^2 - \frac{N}{D} , \quad (4.34)$$

i.e. the right hand sides of (4.28) and (4.33) match. This in turn shows that the right boundary of  $R'_2$  coincides with the left boundary  $L$  of  $R'_1$ , and hence in any region of the moduli space between the two walls of marginal stability  $L_1$  and  $L_2$  in Fig. 2, one and only one of the two configurations exists. This is precisely what is required for the black hole metamorphosis hypothesis to hold.

## 4.6 Special case of diagonal $T^6$

For later use we shall now write down the explicit solutions in the special case

$$\sigma_1 = \rho_1 = 0, \quad (4.35)$$

corresponding to setting the off-diagonal components of the metric and the 2-form field along  $T^2$  to zero at infinity. Furthermore we shall take the location of the first center at the origin so that

$$\vec{r}_1 = 0, \quad |\vec{r} - \vec{r}_1| = r. \quad (4.36)$$

Then we can express (4.15), (4.16) as

$$\begin{aligned} H^0 &= \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \text{sign}(\rho_2 - \sigma_2) \frac{\zeta_1}{|\zeta|}, \\ H^1 &= \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \text{sign}(\rho_2 - \sigma_2) |\zeta|, \\ H^2 &= -\frac{1}{r} + \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \text{sign}(\rho_2 - \sigma_2) \frac{\rho_2\zeta_2}{|\zeta|}, \\ H^3 &= \frac{1}{r} + \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \text{sign}(\rho_2 - \sigma_2) \frac{\sigma_2\zeta_2}{|\zeta|}, \\ H_0 &= -\frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \text{sign}(\rho_2 - \sigma_2) \rho_2\sigma_2|\zeta|, \\ H_1 &= \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \text{sign}(\rho_2 - \sigma_2) \frac{\rho_2\sigma_2\zeta_1}{|\zeta|}, \\ H_2 &= 0, \\ H_3 &= 0. \end{aligned} \quad (4.37)$$

This gives from (4.6), (3.6)-(3.11),

$$\begin{aligned} \Sigma(\{H_I\}, \{H^I\}) &= [-4H_0H^1H^2H^3]^{1/2}, \\ S &= \frac{X^1}{X^0} = \frac{2iH_0H^1}{-\Sigma + 2iH_0H^0}, \quad T = \frac{X^2}{X^0} = -\frac{2iH_0H^2}{\Sigma}, \quad U = \frac{X^3}{X^0} = -\frac{2iH_0H^3}{\Sigma}, \\ ds^2 &= -\Sigma^{-1} dt^2 + \Sigma dx^i dx^i, \\ \mathcal{A}_\mu^0 dx^\mu &= -\frac{2H^1H^2H^3}{\Sigma^2} dt, \quad \mathcal{A}_\mu^2 dx^\mu = -\frac{2H_0H^0H^2}{\Sigma^2} dt + \cos\theta d\phi, \\ \mathcal{A}_\mu^3 dx^\mu &= -\frac{2H_0H^0H^3}{\Sigma^2} dt - \cos\theta d\phi, \quad \mathcal{A}_{1\mu} dx^\mu = \frac{2H_0H^2H^3}{\Sigma^2} dt. \end{aligned} \quad (4.38)$$

Note that we have given the expressions for the electric potentials  $\mathcal{A}_\mu^0, \mathcal{A}_\mu^2, \mathcal{A}_\mu^3$  and the magnetic potential  $\mathcal{A}_{1\mu}$ . This contains full information about all the gauge fields. Eq.(4.38) shows that

$T$  and  $U$  remain purely imaginary for all values of  $\vec{r}$  and hence the off diagonal components of the metric and the 2-form field along  $T^2$  continue to vanish at all points.

From eqs.(3.16) and (4.12) we get the Lagrangian of the test particle carrying charge  $(q_{(2)}, p_{(2)})$  in this background to be

$$\begin{aligned}
L_t &= -\Sigma(\{H_I\}, \{H^I\})^{-1/2} \frac{1}{\sqrt{8S_2(\vec{r})T_2(\vec{r})U_2(\vec{r})}} |a + dT(\vec{r}) + bU(\vec{r}) - cT(\vec{r})U(\vec{r})| \\
&\quad + \frac{1}{2}(c\mathcal{A}_{10} + a\mathcal{A}_0^0 + d\mathcal{A}_0^2 + b\mathcal{A}_0^3) \\
&= -\frac{1}{4H_0} \left\{ 1 + \frac{H^0 H_1}{H^2 H^3} \right\}^{1/2} \left[ \left( a - c \frac{H_0}{H^1} \right)^2 - \frac{H_0}{H^1 H^2 H^3} (dH^2 + bH^3)^2 \right]^{1/2} \\
&\quad - \frac{1}{4} \left\{ -\frac{a}{H_0} + d \frac{H_1}{H_0 H^3} + b \frac{H_1}{H_0 H^2} + \frac{c}{H^1} \right\}. \tag{4.39}
\end{aligned}$$

The equilibrium separation (4.17) between the two centers can be found by extremizing (4.39) with respect to  $r$ . Corresponding result for the second system is obtained by exchanging  $b$  and  $d$  in (4.39).

## 5 Replacing the enhancon by the smooth solution

In this section we shall replace the solution in the S-T-U model described in §4.6 by a smooth dyon solution and compute the range of values of  $\zeta_1$  for which the solution exists. Since the analysis of this section will be somewhat technical let us first summarize the main result. We shall find that the net effect of smoothening the solution is to replace in the expressions for  $\Sigma$ ,  $S_2$ ,  $T$ ,  $U$ ,  $\mathcal{A}_0^0$ ,  $\mathcal{A}_0^2$ ,  $\mathcal{A}_0^3$  and  $\mathcal{A}_{10}$  given in (4.38), the variable  $r$  by  $\hat{r}$  where

$$\frac{1}{\hat{r}} = \frac{1}{r} - \kappa \coth(\kappa r) + \kappa, \quad \kappa \equiv \sqrt{\frac{\zeta_2}{8\rho_2\sigma_2} \frac{|\rho_2 - \sigma_2|}{|\zeta|}} = \frac{1}{r_e}. \tag{5.1}$$

This does not mean that the new solution is related to the old one by a coordinate transformation since for example the  $dx^i dx^i$  term in the expression for the metric is still given by  $dr^2 + r^2 d\Omega_2^2$  with  $d\Omega_2$  denoting the line element on a unit 2-sphere. Nevertheless it shows that the potential for the test charge in this new background is given by (4.39) with  $r$  replaced by  $\hat{r}$  everywhere. Thus for given values of the asymptotic moduli the equilibrium position of the test charge  $(Q, 0)$  or  $(Q + uP, 0)$  is given by replacing  $|\vec{r}_1 - \vec{r}_2| = |\vec{r}_2|$  by  $\hat{r}_2$  in the S-T-U model results (4.17) and (4.29) respectively, where  $\hat{r}_2$  is the value of  $\hat{r}$  defined in (5.1) for  $r = |\vec{r}_2|$ .<sup>1</sup>

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<sup>1</sup>We are again setting  $\vec{r}_1 = 0$  i.e. taking the location of the first center as the origin of the coordinate system.

Now since according to (5.1)  $r = 0$  corresponds to  $\hat{r} = r_e$  and  $r = \infty$  corresponds to  $\hat{r} = \infty$  we see that requiring  $0 < |\vec{r}_2| < \infty$  corresponds to  $r_e < \hat{r}_2 < \infty$ . This according to the analysis of §4 (with  $r$  replaced by  $\hat{r}$ ) constraints  $\tau$  to lie inside the region  $R'_1$  of Fig. 2 for the first configuration and the region  $R'_2$  of Fig. 2 for the second configuration. Thus we conclude that the ranges of  $\tau_1$  for which the solutions exist remain the same as what we derived in §4. However the interpretation of what happens at the left boundary of  $R'_1$  and the right boundary of  $R'_2$  is slightly different. At the left boundary of  $R'_1$ , when  $\tau_1$  saturates the bound (4.28), the second center of the first configuration reaches the center of the smooth dyonic solution. On the other hand at the right boundary of  $R'_2$ , when  $\tau_1$  saturates the bound (4.33), the second center of the second configuration reaches the center of the smooth dyonic solution.

We shall now describe how these results arise.

## 5.1 Harvey-Liu monopole and dyon solutions in the ten dimensional description

We shall consider a truncation of the effective action of ten dimensional heterotic string theory where we keep only a single  $SU(2)$  gauge field  $\mathcal{V}_\mu^{(a)}$  ( $1 \leq a \leq 3$ ) out of  $SO(32)$  or  $E_8 \times E_8$ . This action is given by

$$S = \frac{2\pi}{(2\pi\sqrt{\alpha'})^8} \int d^{10}x \sqrt{-\det G} e^{-2\Phi} \left[ R + 4G^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{12} G^{MM'} G^{NN'} G^{RR'} H_{MNNR} H_{M'N'R'} - \frac{\alpha'}{8} \mathcal{W}_{MN}^{(a)} \mathcal{W}^{(a)MN} \right],$$

$$\mathcal{W}_{MN}^{(a)} \equiv \partial_M \mathcal{V}_N^{(a)} - \partial_N \mathcal{V}_M^{(a)} + \epsilon^{abc} \mathcal{V}_M^{(b)} \mathcal{V}_N^{(c)}, \quad (5.2)$$

$$dH = -\frac{\alpha'}{4} \mathcal{W}^{(a)} \wedge \mathcal{W}^{(a)}, \quad H \equiv \frac{1}{3!} H_{MNP} dx^M \wedge dx^N \wedge dx^P, \quad \mathcal{W}^{(a)} \equiv \frac{1}{2!} \mathcal{W}_{MN}^{(a)} dx^M \wedge dx^N. \quad (5.3)$$

Here  $x^M$  for  $0 \leq M \leq 9$  are the coordinates labelling the ten dimensional space-time,  $G_{MN}$  is the string metric,  $H$  is the 3-form field strength and  $\Phi$  is the dilaton field. We now compactify the theory on  $T^6$  labelled by  $x^4, \dots, x^9$  with period  $2\pi\sqrt{\alpha'}$  and non-compact coordinates labelled

by  $x^0, x^1, x^2, x^3$ . In this theory we consider the Harvey-Liu monopole solution [27, 28]<sup>2</sup>

$$\begin{aligned}
\mathcal{V}_i^{(a)} &= \epsilon_{iak} \frac{x^k}{r^2} (K(C_1 r) - 1), \quad \mathcal{V}_4^{(a)} = C_2 \frac{x^a}{r^2} H(C_1 r), \quad 1 \leq i, k, a \leq 3, \quad r \equiv \sqrt{x^k x^k}, \\
H(x) &\equiv x \coth x - 1, \quad K(x) = x / \sinh x, \\
e^{2\Phi} &= C_3^2 + \frac{\alpha'}{4} (C_1^2 - r^{-2} H(C_1 r)^2), \\
ds^2 &= -(dx^0)^2 + e^{2\Phi} ((dx^1)^2 + (dx^2)^2 + (dx^3)^2 + C_2^2 (dx^4)^2) + C_4^2 (dx^5)^2 + \sum_{m=6}^9 dx^m dx^m, \\
H_{4ij} &= -2 C_2 e^{2\Phi} \epsilon_{ijk} \partial_k \Phi \quad 1 \leq i, j, k \leq 3,
\end{aligned} \tag{5.4}$$

where  $C_1, C_2, C_3$  and  $C_4$  are arbitrary constants, and  $\epsilon_{ijk}$  is the totally anti-symmetric symbol with  $\epsilon_{123} = 1$ . Since all the fields in (5.4) are invariant under changes in signs of  $C_1, C_3$  and  $C_4$ , we can choose

$$C_1, C_4, C_2 C_3 > 0, \tag{5.5}$$

without any loss of generality. Note that the solution described in (5.4) lies outside the truncated theory described in §4.1 since we have non-trivial background values of the ten dimensional gauge fields. However we shall see that (the dyonic generalization of) this solution can be mapped to a solution inside the truncated theory by a duality rotation.

Physically (5.4) represents a gravitationally dressed BPS monopole solution of the  $SU(2)$  gauge theory. We can construct from this a dyon solution by making the replacement (see *e.g.* [29])

$$x^0 \rightarrow \cosh \gamma x^0 + C_2 C_3 \sinh \gamma x^4, \quad x^4 \rightarrow C_2^{-1} C_3^{-1} \sinh \gamma x^0 + \cosh \gamma x^4, \tag{5.6}$$

and taking the new  $x^4$  coordinate defined this way as being periodically identified with period  $2\pi\sqrt{\alpha'}$ . This gives a solution:

$$\begin{aligned}
\mathcal{V}_i^{(a)} &= \epsilon_{iak} \frac{x^k}{r^2} (K(C_1 r) - 1), \quad \mathcal{V}_4^{(a)} = C_2 \cosh \gamma \frac{x^a}{r^2} H(C_1 r), \\
\mathcal{V}_0^{(a)} &= C_3^{-1} \sinh \gamma \frac{x^a}{r^2} H(C_1 r),
\end{aligned}$$

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<sup>2</sup>Strictly speaking, if we take the circles labelled by  $x^6, \dots, x^9$  to have self-dual radius, as is the case for the metric given in (5.4), we shall get additional massless non-abelian gauge fields. We can avoid this situation by taking the metric along the  $x^6, \dots, x^9$  direction to be  $K_{mn} dx^m dx^n$  for some constant symmetric matrix  $K$  with  $\det K = 1$ . This does not affect any of the subsequent analysis. Similarly we could also break the rest of the ten dimensional gauge group ( $SO(32)$  or  $E_8 \times E_8$ ) by turning on Wilson lines for these gauge fields along the 6-7-8-9 directions without changing any of the subsequent analysis.

$$\begin{aligned}
e^{2\Phi} &= C_3^2 + \frac{\alpha'}{4}(C_1^2 - r^{-2}H(C_1r)^2), \\
ds^2 &= -(dx^0)^2 + e^{2\Phi}((dx^1)^2 + (dx^2)^2 + (dx^3)^2) + C_2^2 C_3^2 (dx^4)^2 + C_4^2 (dx^5)^2 + \sum_{m=6}^9 dx^m dx^m \\
&\quad + (e^{2\Phi} C_3^{-2} - 1) (\sinh \gamma dx^0 + C_2 C_3 \cosh \gamma dx^4)^2, \\
H_{4ij} &= -2 C_2 \cosh \gamma e^{2\Phi} \epsilon_{ijk} \partial_k \Phi, \\
H_{0ij} &= -2 C_3^{-1} \sinh \gamma e^{2\Phi} \epsilon_{ijk} \partial_k \Phi, \quad 1 \leq i, j, k, a \leq 3.
\end{aligned} \tag{5.7}$$

The solutions given above are in the hedgehog gauge. For comparison with the solution in the S-T-U model it will be more appropriate to express the solution in the string gauge (see *e.g.* [28]). In this gauge the solution takes the form

$$\begin{aligned}
\mathcal{V}_i^{(3)} dx^i &\simeq \cos \theta d\phi, \\
\mathcal{V}_4^{(3)} &= C_2 \cosh \gamma \frac{1}{r} H(C_1 r), \\
\mathcal{V}_0^{(3)} &= C_3^{-1} \sinh \gamma \frac{1}{r} H(C_1 r) \\
e^{2\Phi} &= C_3^2 + \frac{\alpha'}{4}(C_1^2 - r^{-2}H(C_1r)^2) \\
ds^2 &= -(dx^0)^2 + e^{2\Phi}((dx^1)^2 + (dx^2)^2 + (dx^3)^2) + C_2^2 C_3^2 (dx^4)^2 + C_4^2 (dx^5)^2 + \sum_{m=6}^9 dx^m dx^m \\
&\quad + (e^{2\Phi} C_3^{-2} - 1) (\sinh \gamma dx^0 + C_2 C_3 \cosh \gamma dx^4)^2 \\
H_{4ij} &= -2 C_2 \cosh \gamma e^{2\Phi} \epsilon_{ijk} \partial_k \Phi, \\
H_{0ij} &= -2 C_3^{-1} \sinh \gamma e^{2\Phi} \epsilon_{ijk} \partial_k \Phi, \quad 1 \leq i, j, k \leq 3.
\end{aligned} \tag{5.8}$$

The  $\simeq$  in the first equation describes equality up to terms of order  $e^{-C_1 r}$  and also additive constants.

From now on we shall work in the  $\alpha' = 16$  unit. For reason that will become clear later, we shall choose the constants  $C_i$ 's and  $\gamma$  such that

$$G_{44} + 4(\mathcal{V}_4^{(3)})^2 = C_2^2 C_3^2 + 4C_1^2 C_2^2 \cosh^2 \gamma = 1. \tag{5.9}$$

## 5.2 Smooth dyon solution in the four dimensional description

We now translate the above solution into a field configuration in an effective four dimensional field theory. For this we dimensionally reduce the theory to four dimensions, keeping a single

$U(1)$  gauge field  $\mathcal{V}_M^{(3)}$  in ten dimensions, and setting the components of various fields along  $T^4$ , labelled by the coordinates  $x^6, \dots, x^9$ , to their background values given in (5.8), and setting  $\alpha' = 16$ . This leads to an action whose bosonic part is given by:

$$S = \frac{1}{32\pi} \int d^4x \sqrt{-\det g} \left[ R - \frac{1}{2S_2^2} g^{\mu\nu} \partial_\mu S \partial_\nu \bar{S} - S_2 F_{\mu\nu}^{(a)} (LML)_{ab} F^{(b)\mu\nu} + S_1 F_{\mu\nu}^{(a)} L_{ab} \tilde{F}^{(b)\mu\nu} + \frac{1}{8} g^{\mu\nu} \text{Tr}(\partial_\mu ML \partial_\nu ML) \right]. \quad (5.10)$$

Here  $S = S_1 + iS_2$  is a complex scalar field representing the heterotic axion - dilaton system,  $F_{\mu\nu}^{(a)} \equiv \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)}$  for  $1 \leq a \leq 5$  are the gauge field strengths associated with five  $U(1)$  gauge fields  $A_\mu^{(a)}$ ,  $\tilde{F}_{\mu\nu}$  denotes the dual field strength of  $F_{\mu\nu}$ ,  $L$  is the  $5 \times 5$  matrix

$$L = \begin{pmatrix} 0 & I_2 & \\ I_2 & 0 & \\ & & -1 \end{pmatrix}, \quad (5.11)$$

with  $I_n$  denoting  $n \times n$  identity matrix, and  $M$  is a matrix valued scalar field, satisfying

$$MLM^T = L, \quad M^T = M. \quad (5.12)$$

The precise relation between the fields appearing here and those in the ten dimensional supergravity was given in [30] and reviewed in [31].<sup>3</sup> We shall use the normalization convention of [31], keeping in mind that  $\mathcal{V}_\mu^{(3)}$  is related to the ten dimensional abelian gauge fields  $A_\mu^{(10)I}$  used in [31] as  $A_\mu^{(10)1} = 2\sqrt{2}\mathcal{V}_\mu^{(3)}$ . In order to facilitate comparison with the fields of the S-T-U model as reviewed in §4, where the normalization in front of the Einstein-Hilbert term is given by  $1/16\pi$ , we shall make a  $g_{\mu\nu} \rightarrow 2g_{\mu\nu}$  field redefinition, so that the action takes the form:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-\det g} \left[ R - \frac{1}{2S_2^2} g^{\mu\nu} \partial_\mu S \partial_\nu \bar{S} - \frac{1}{2} S_2 F_{\mu\nu}^{(a)} (LML)_{ab} F^{(b)\mu\nu} + \frac{1}{2} S_1 F_{\mu\nu}^{(a)} L_{ab} \tilde{F}^{(b)\mu\nu} + \frac{1}{8} g^{\mu\nu} \text{Tr}(\partial_\mu ML \partial_\nu ML) \right]. \quad (5.13)$$

If we denote the metric appearing in (5.8) by  $G_{MN}$  and define

$$A_4 = 2\sqrt{2}\mathcal{V}_4^{(3)}, \quad (5.14)$$

then using the results reviewed in [31] we find that the four dimensional field configuration

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<sup>3</sup>In the convention of [31] that we shall use,  $S$  corresponds to the field  $\lambda$ .

corresponding to the background (5.8) is given by<sup>4</sup>

$$\begin{aligned}
M &= \begin{pmatrix} G_{55}^{-1} & 0 & 0 & 0 & 0 \\ 0 & G_{44}^{-1} & 0 & \frac{1}{2}G_{44}^{-1}A_4^2 & G_{44}^{-1}A_4 \\ 0 & 0 & G_{55} & 0 & 0 \\ 0 & \frac{1}{2}G_{44}^{-1}A_4^2 & 0 & (G_{44} + \frac{1}{2}A_4^2)^2G_{44}^{-1} & (G_{44} + \frac{1}{2}A_4^2)G_{44}^{-1}A_4 \\ 0 & G_{44}^{-1}A_4 & 0 & (G_{44} + \frac{1}{2}A_4^2)G_{44}^{-1}A_4 & 1 + G_{44}^{-1}A_4^2 \end{pmatrix} \\
&= \begin{pmatrix} G_{55}^{-1} & 0 & 0 & 0 & 0 \\ 0 & G_{44}^{-1} & 0 & \frac{1}{2}G_{44}^{-1}A_4^2 & G_{44}^{-1}A_4 \\ 0 & 0 & G_{55} & 0 & 0 \\ 0 & \frac{1}{2}G_{44}^{-1}A_4^2 & 0 & G_{44}^{-1} & G_{44}^{-1}A_4 \\ 0 & G_{44}^{-1}A_4 & 0 & G_{44}^{-1}A_4 & 1 + G_{44}^{-1}A_4^2 \end{pmatrix}, \tag{5.15}
\end{aligned}$$

where in the last step we have used (5.9),

$$S_2 = e^{-2\Phi} C_2 C_4 \sqrt{e^{2\Phi} \cosh^2 \gamma - C_3^2 \sinh^2 \gamma}, \tag{5.16}$$

$$S_1 \simeq C_2 C_3 C_4 \sinh \gamma e^{-2\Phi}, \tag{5.17}$$

$$\begin{aligned}
\{A_0^{(a)}\} &= -\sqrt{2} C_3 \sinh \gamma (e^{2\Phi} \cosh^2 \gamma - C_3^2 \sinh^2 \gamma)^{-1} \begin{pmatrix} 0 \\ -\frac{1}{2\sqrt{2}} C_2^{-1} \cosh \gamma (e^{2\Phi} C_3^{-2} - 1) \\ 0 \\ \sqrt{2} C_2 \cosh \gamma r^{-2} H(C_1 r)^2 \\ r^{-1} H(C_1 r) \end{pmatrix}, \\
\{A_i^{(a)} dx^i\} &= -\sqrt{2} \cos \theta d\phi \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{5.18}
\end{aligned}$$

$$g_{\mu\nu} dx^\mu dx^\nu = -\frac{C_2 C_4}{2\sqrt{e^{2\Phi} \cosh^2 \gamma - C_3^2 \sinh^2 \gamma}} (dx^0)^2 + \frac{1}{2} C_2 C_4 \sqrt{e^{2\Phi} \cosh^2 \gamma - C_3^2 \sinh^2 \gamma} dx^i dx^i. \tag{5.19}$$

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<sup>4</sup>In order to get the expression for  $S_1$  given in (5.17), we need to correct the formula for the 4-dimensional 2-form field  $B_{\mu\nu}$  given in eq.(3) of [31]. The corrected expression is given by

$$B_{\mu\nu} = B_{\mu\nu}^{(10)} - 4\widehat{B}_{mn} A_\mu^{(m)} A_\nu^{(n)} - 2 \left( A_\mu^{(m)} A_\nu^{(m+6)} - A_\nu^{(m)} A_\mu^{(m+6)} \right) - 2\widehat{A}_m^I \left( A_\mu^{(I+12)} A_\nu^{(m)} - A_\nu^{(I+12)} A_\mu^{(m)} \right)$$

in the notation of [31]. The last term was missed in [31] but is needed to ensure that  $B_{\mu\nu}$  transforms correctly under the gauge transformation of  $A_\mu^{(I+12)}$ .

We now take the  $5 \times 5$  matrix

$$\begin{aligned}
W &\equiv \begin{pmatrix} I_2/\sqrt{2} & I_2/\sqrt{2} & & & \\ I_2/\sqrt{2} & -I_2/\sqrt{2} & & & \\ & & 1 & & \\ & & & I_3 & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} \begin{pmatrix} I_3 & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \end{pmatrix} \begin{pmatrix} I_2/\sqrt{2} & I_2/\sqrt{2} & & & \\ I_2/\sqrt{2} & -I_2/\sqrt{2} & & & \\ & & & & & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{5.20}
\end{aligned}$$

satisfying

$$W^T W = I_5, \quad W^T L W = L, \tag{5.21}$$

and make the field redefinition:

$$M \rightarrow W M W^T, \quad F_{\mu\nu}^{(a)} \rightarrow W_{ab} F_{\mu\nu}^{(b)}. \tag{5.22}$$

The action in the new variables takes the same form as (5.10). After this transformation the solution (5.15) for  $M$  becomes

$$M = \begin{pmatrix} \tilde{R}^{-2} & 0 & 0 & 0 & 0 \\ 0 & R^{-2} & 0 & 0 & 0 \\ 0 & 0 & \tilde{R}^2 & 0 & 0 \\ 0 & 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.23}$$

where

$$\tilde{R}^2 = G_{55} = C_4^2, \quad R^2 = \frac{1 - \frac{1}{\sqrt{2}} A_4}{1 + \frac{1}{\sqrt{2}} A_4} = \frac{1 - 2\mathcal{V}_4^{(3)}}{1 + 2\mathcal{V}_4^{(3)}} = \frac{1 - 2C_2 \cosh \gamma r^{-1} H(C_1 r)}{1 + 2C_2 \cosh \gamma r^{-1} H(C_1 r)}. \tag{5.24}$$

The gauge field background takes the form, up to constant shifts,

$$\begin{aligned}
\{A_0^{(a)}\} &= C_3 C_2^2 \sinh \gamma \frac{1}{r} H(C_1 r) \begin{pmatrix} 0 \\ \{2C_2 \cosh \gamma r^{-1} H(C_1 r) - 1\}^{-1} \\ 0 \\ \{2C_2 \cosh \gamma r^{-1} H(C_1 r) + 1\}^{-1} \\ 0 \end{pmatrix}, \\
\{A_i^{(a)} dx^i\} &\simeq \cos \theta d\phi \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{5.25}
\end{aligned}$$

The metric and the axion-dilaton fields remain unchanged under this field redefinition.

We now note that for the solution described above the matrix  $M$  and the gauge fields are non-trivial only along the first four rows and columns. This corresponds to setting to zero all ten dimensional gauge fields and also setting all components of the metric and 2-form fields with one or both legs along  $T^4$  to trivial values. This is precisely the condition under which the solution can be embedded in the S-T-U model. Rescaling  $x^i$  and  $x^0$  as

$$x^i \rightarrow \sqrt{\frac{2}{C_2 C_3 C_4}} x^i, \quad x^0 \rightarrow x^0 \sqrt{\frac{2C_3}{C_2 C_4}}, \quad (5.26)$$

and identifying  $R\tilde{R}$  with  $T_2$  and  $\tilde{R}/R$  with  $U_2$  we see that in the variables of the S-T-U model the scalar fields and the metric takes the form:

$$\begin{aligned} T_1 &= 0, & U_1 &= 0, \\ T_2 U_2 &= C_4^2, & \frac{T_2}{U_2} &= \frac{1 - C_2 \cosh \gamma \sqrt{2 C_2 C_3 C_4} r^{-1} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4})}{1 + C_2 \cosh \gamma \sqrt{2 C_2 C_3 C_4} r^{-1} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4})}, \\ S_2 &= e^{-2\Phi} C_2 C_4 \sqrt{e^{2\Phi} \cosh^2 \gamma - C_3^2 \sinh^2 \gamma}, \\ S_1 &\simeq C_2 C_3 C_4 \sinh \gamma e^{-2\Phi}, \\ g_{\mu\nu} dx^\mu dx^\nu &= -e^{2V} (dx^0)^2 + e^{-2V} dx^i dx^i, \\ e^{2\Phi} &= C_3^2 + 4 \left( C_1^2 - \frac{C_2 C_3 C_4}{2r^2} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4})^2 \right), & e^{2V} &\equiv \frac{C_3}{\sqrt{e^{2\Phi} \cosh^2 \gamma - C_3^2 \sinh^2 \gamma}}. \end{aligned} \quad (5.27)$$

To find the gauge fields in the S-T-U model notation we first note that after the coordinate change (5.26) the first four components of gauge fields  $A_\mu^{(a)}$  given in (5.25) takes the form

$$\begin{aligned} \{A_0^{(a)}\} &= C_3^2 C_2^2 \sinh \gamma \frac{1}{r} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4}) \\ &\quad \left( \begin{array}{c} 0 \\ \left\{ 2C_2 \cosh \gamma \sqrt{\frac{C_2 C_3 C_4}{2}} r^{-1} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4}) - 1 \right\}^{-1} \\ 0 \\ \left\{ 2C_2 \cosh \gamma \sqrt{\frac{C_2 C_3 C_4}{2}} r^{-1} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4}) + 1 \right\}^{-1} \end{array} \right), \\ \{A_i^{(a)} dx^i\} &\simeq \cos \theta d\phi \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (5.28)$$

Now it was shown in [31] that a test charge  $(Q, 0)$  couples to this gauge field background through the action

$$\begin{aligned} \pm \frac{1}{2} \int dx^\mu A_\mu^{(a)} Q_a &= \pm \frac{1}{2} \int dx^\mu [A_\mu^{(1)} Q_1 + A_\mu^{(2)} Q_2 + A_\mu^{(3)} Q_3 + A_\mu^{(4)} Q_4] \\ &= \pm \frac{1}{2} \int dx^\mu [A_\mu^{(1)} q_0 + A_\mu^{(2)} q_3 - A_\mu^{(3)} p^1 + A_\mu^{(4)} q_2] . \end{aligned} \quad (5.29)$$

The  $\pm$  sign reflects the fact that the analysis of [31] determines the normalization but not the sign of the coupling of the gauge fields to the charges since the bosonic action involving the  $U(1)$  gauge fields has an  $A_\mu \rightarrow -A_\mu$  symmetry. Comparing this with (3.2) we get

$$\begin{pmatrix} \mathcal{A}_\mu^0 \\ \mathcal{A}_\mu^3 \\ \mathcal{A}_{1\mu} \\ \mathcal{A}_\mu^2 \end{pmatrix} = \pm \begin{pmatrix} A_\mu^{(1)} \\ A_\mu^{(2)} \\ A_\mu^{(3)} \\ A_\mu^{(4)} \end{pmatrix} . \quad (5.30)$$

Eq.(5.28) now shows that the magnetic part of the field is given by

$$\mathcal{A}_i^3 dx^i \simeq \mp \cos \theta d\phi, \quad \mathcal{A}_i^2 dx^i = \pm \cos \theta dx^i . \quad (5.31)$$

On the other hand (4.38) shows that the expected magnetic field in the S-T-U model, produced by the first center, is given by

$$\mathcal{A}_i^3 dx^i = -\cos \theta d\phi, \quad \mathcal{A}_i^2 dx^i = \cos \theta dx^i . \quad (5.32)$$

Comparing (5.31) and (5.32) we see that we must use the top sign in (5.30). This can now be used to express the electric potentials given in (5.28) as

$$\begin{aligned} \begin{pmatrix} \mathcal{A}_0^0 \\ \mathcal{A}_0^3 \\ \mathcal{A}_{10} \\ \mathcal{A}_0^2 \end{pmatrix} &= \begin{pmatrix} A_0^{(1)} \\ A_0^{(2)} \\ A_0^{(3)} \\ A_0^{(4)} \end{pmatrix} = C_3^2 C_2^2 \sinh \gamma \frac{1}{r} H \left( \sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4} \right) \\ &\quad \begin{pmatrix} 0 \\ \left\{ 2C_2 \cosh \gamma \sqrt{\frac{C_2 C_3 C_4}{2}} r^{-1} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4}) - 1 \right\}^{-1} \\ 0 \\ \left\{ 2C_2 \cosh \gamma \sqrt{\frac{C_2 C_3 C_4}{2}} r^{-1} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4}) + 1 \right\}^{-1} \end{pmatrix} . \end{aligned} \quad (5.33)$$

Finally, adding constant terms to the gauge potential, we can bring (5.33) to the form:

$$\begin{pmatrix} \mathcal{A}_0^0 \\ \mathcal{A}_0^3 \\ \mathcal{A}_{10} \\ \mathcal{A}_0^2 \end{pmatrix} = C_3^2 C_2^2 \sinh \gamma \frac{1}{2C_2 \cosh \gamma} \sqrt{\frac{2}{C_2 C_3 C_4}} \begin{pmatrix} 0 \\ \left\{ 2C_2 \cosh \gamma \sqrt{\frac{C_2 C_3 C_4}{2}} r^{-1} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4}) - 1 \right\}^{-1} \\ 0 \\ - \left\{ 2C_2 \cosh \gamma \sqrt{\frac{C_2 C_3 C_4}{2}} r^{-1} H(\sqrt{2} C_1 r / \sqrt{C_2 C_3 C_4}) + 1 \right\}^{-1} \end{pmatrix}. \quad (5.34)$$

Defining

$$\frac{1}{\hat{r}} = \kappa - \frac{1}{r} H(\kappa r) = \frac{1}{r} - \kappa \coth(\kappa r) + \kappa, \quad (5.35)$$

where

$$\kappa = \frac{\sqrt{2} C_1}{\sqrt{C_2 C_3 C_4}}, \quad (5.36)$$

we can express (5.34), (5.27) as

$$\begin{pmatrix} \mathcal{A}_0^0 \\ \mathcal{A}_0^3 \\ \mathcal{A}_{10} \\ \mathcal{A}_0^2 \end{pmatrix} = \frac{1}{2} \frac{C_3}{C_2 C_4} \frac{\sinh \gamma}{\cosh^2 \gamma} \begin{pmatrix} 0 \\ \left\{ -\frac{1-2C_1 C_2 \cosh \gamma}{2C_1 C_2 \cosh \gamma} \kappa - \frac{1}{\hat{r}} \right\}^{-1} \\ 0 \\ \left\{ -\frac{1+2C_1 C_2 \cosh \gamma}{2C_1 C_2 \cosh \gamma} \kappa + \frac{1}{\hat{r}} \right\}^{-1} \end{pmatrix},$$

$$T_1 = 0, \quad U_1 = 0,$$

$$T_2 U_2 = C_4^2, \quad \frac{T_2}{U_2} = \frac{1 - C_2 \cosh \gamma \sqrt{2 C_2 C_3 C_4} (\kappa - \hat{r}^{-1})}{1 + C_2 \cosh \gamma \sqrt{2 C_2 C_3 C_4} (\kappa - \hat{r}^{-1})}$$

$$S_2 = e^{-2\Phi} C_2 C_4 \sqrt{e^{2\Phi} \cosh^2 \gamma - C_3^2 \sinh^2 \gamma},$$

$$S_1 \simeq C_2 C_3 C_4 \sinh \gamma e^{-2\Phi},$$

$$g_{\mu\nu} dx^\mu dx^\nu = -e^{2V} (dx^0)^2 + e^{-2V} dx^i dx^i,$$

$$e^{2\Phi} = C_3^2 + 4 \left( C_1^2 - \frac{C_2 C_3 C_4}{2} (\kappa - \hat{r}^{-1})^2 \right), \quad e^{2V} \equiv \frac{C_3}{\sqrt{e^{2\Phi} \cosh^2 \gamma - C_3^2 \sinh^2 \gamma}}. \quad (5.37)$$

For large  $r$  we have  $H(r) \simeq r-1$  and hence  $\hat{r} \simeq r$  up to exponentially suppressed corrections. In that case the field configurations given in (5.37) agree with those given in (4.38) (up to constant additive terms in the gauge potential) with the choice

$$\rho_2 = C_4 \sqrt{\frac{1 - 2C_1 C_2 \cosh \gamma}{1 + 2C_1 C_2 \cosh \gamma}}, \quad \sigma_2 = C_4 \sqrt{\frac{1 + 2C_1 C_2 \cosh \gamma}{1 - 2C_1 C_2 \cosh \gamma}},$$

$$\zeta_2 = \frac{C_2 C_4}{C_3}, \quad \zeta_1 = \frac{C_2 C_4}{C_3} \sinh \gamma. \quad (5.38)$$

Under this identification,  $\kappa$  given in (5.36) becomes

$$\kappa = \sqrt{\frac{\zeta_2}{8\rho_2\sigma_2} \frac{|\rho_2 - \sigma_2|}{|\zeta|}} = \frac{1}{r_e}. \quad (5.39)$$

Now note that for finite  $r$  the solutions for  $S_2, T, U, V$  and  $\mathcal{A}_0^I, \mathcal{A}_{I0}$  are given by the same expressions as in the case of S-T-U model described in §4 with the replacement of  $r$  by  $\hat{r}$ . Since these are the fields which determine the location of the test particle charge (by the extrema of (4.39)), we can directly take the results of section 4 with  $r$  replaced by  $\hat{r}$  for determining the location of the test charge. Now from (5.35) we see that the condition  $r > 0$  corresponds to  $\hat{r} > 1/\kappa = r_e$ . Thus requiring  $|\vec{r}_2|$  to be positive corresponds to requiring  $\hat{r}_2$ , – the value of  $\hat{r}$  corresponding to the vector  $\vec{r}_2$  – be larger than  $r_e$ . On the other hand for large  $r$  we have  $r \simeq \hat{r}$ . Thus the condition  $0 < |\vec{r}_2| < \infty$  translates to  $r_e \leq \hat{r}_2 < \infty$ . Since we can use the results of §4 for determining the location of the test charge with  $r$  replaced by  $\hat{r}$ , we see that the condition  $r_e \leq \hat{r}_2 < \infty$  translates to requiring  $\tau_1$  to lie inside the range given in (4.24), (4.28) for the first configuration and inside the range given in (4.32), (4.33) for the second configuration. These two ranges do not overlap, and together they make up the region  $R'_1 \cup R'_2$  of the moduli space shown in Fig. 2 – precisely in agreement with the microscopic result for the index.

This still leaves open the question as to how the two configurations metamorphose into each other at the boundary  $L$  of  $R'_1$  and  $R'_2$ . To examine this we apply the inverse of the duality

transformation (5.20) to map the test electric charges  $Q = \begin{pmatrix} a \\ b \\ c \\ d \\ 0 \end{pmatrix}$  and  $Q + uP = \begin{pmatrix} a \\ d \\ c \\ b \\ 0 \end{pmatrix}$  to

$$\begin{pmatrix} a \\ (b+d)/2 \\ c \\ (b+d)/2 \\ (b-d)/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a \\ (b+d)/2 \\ c \\ (b+d)/2 \\ (d-b)/\sqrt{2} \end{pmatrix}. \quad (5.40)$$

The last entry represents electric charge under the  $T^3$  generator of the SU(2) group. Now at the center of the dyon solution the SU(2) gauge symmetry is restored. Thus at no cost in energy, the test electric charge can undergo an SU(2) rotation of  $\pi$  about the 1-axis flipping the sign

of the  $T^3$  charge. This exchanges the quantum number  $b$  and  $d$ , precisely transforming the test electric charges of the two configurations to each other. Thus we see that the two configurations can transform into each other at the boundary  $L$  between  $R'_1$  and  $R'_2$ . The excess charge  $-uP$  is dumped into the background, but we do not detect it in the probe approximation that we are using.

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