

Structure formation in the presence of relativistic heat conduction: corrections to the Jeans wave number with a stable first order in the gradients formalism

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Abstract

The problem of structure formation in relativistic dissipative fluids was analyzed in a previous work within Eckart's framework, in which the heat flux is coupled to the hydrodynamic acceleration, additional to the usual temperature gradient term. It was shown that in such case, the pathological behavior of fluctuations leads to the disappearance of the gravitational instability responsible for structure formation [1]. In the present work the problem is revisited now using a constitutive equation derived from relativistic kinetic theory. The new relation, in which the heat flux is not coupled to the hydrodynamic acceleration, leads to a consistent first order in the gradients formalism. In this case the gravitational instability remains, and only relativistic corrections to the Jeans wave number are obtained. In the calculation here shown the non-relativistic limit is recovered, opposite to what happens in Eckart's case [2].

I. INTRODUCTION

Relativistic hydrodynamics has been a subject of interest since the 1940's. However, many open questions remain unanswered, in particular the nature of heat conduction in hot gases has recently been under research and debate. The main issue in such a subject concerns the particular structure of the constitutive equation for dissipative energy flow in the special relativistic case. On the one hand, C. Eckart in 1940, proposed a relation which coupled heat to the hydrodynamic acceleration [3]. This relation was obtained phenomenologically as a sufficient condition for the entropy production to remain positive. Later in 1985, Hiscock and Lindblom working relativistic dissipative fluids, found generic instabilities in Eckart's formalism which in part lead to the development and use of extended theories [2]. A subsequent study showed that these instabilities are due to the presence of the hydrodynamic acceleration in the heat flux constitutive equation [4].

On the other hand several authors have found that kinetic theory predicts a coupling of heat with gradients of state variables instead of the acceleration [5–7]. At the present time, both type of constitutive equations have been used in order to study the Rayleigh-Brillouin scattering spectrum, finding that by introducing Eckart's equation the results were inconsistent with the observations, while the use of the equations in terms of gradients eliminates the inconsistencies [8].

In recent work, two of us addressed the question of structure formation in relativistic dilute gases in the presence of dissipation introducing Eckart's equation for the heat flux [1]. The results obtained were similar to those presented by Hiscock & Lindblom, that is the appearance of an instability of early onset triggered by the fluid itself. This instability, additional to not being observed, rules out any possibility of structure formation in such gases since the growth of density fluctuations due to the gravitational field are overridden.

In the present work, we instead use the constitutive equation obtained in relativistic kinetic theory in the study of structure formation and find that generic instabilities are eliminated in such a way so that gravity becomes dominant. We finally establish the relativistic corrections to the critical wave number.

We have structured this paper as follows. In Sect. II we establish the system of relativistic transport equations for a gas including gravitational effects within the Newtonian limit and linearize around equilibrium values for the state variables in order to obtain a set of linear

equations for the fluctuations. In Sect. III we calculate the dispersion relation for the system in the Fourier-Laplace space and infer from its solutions the behavior of density fluctuations. Section IV is devoted to the explicit calculation of the relativistic correction to the Jeans wave number in a dissipative medium and compare the result with its non-relativistic counterpart. Conclusions and final remarks are included in Sect. V.

II. LINEARIZED TRANSPORT EQUATIONS

The behavior of density fluctuations in a fluid is analyzed by studying the response of the corresponding linearized system of transport equations when the state variables are perturbed from their equilibrium values [9]. That is, if X is a state variable one considers

$$X = X_0 + \delta X \quad (1)$$

where X_0 is the equilibrium value and δX the fluctuation. The gas here considered is a dilute cloud of particles with a temperature high enough, so that the relativistic effects in the molecular motion is relevant. We consider the temperature T , number density n and hydrodynamic four-velocity u^μ as the independent, local equilibrium, state variables. The system of equations where the hypothesis in Eq. (1) is to be introduced is composed of two tensor conservation equations namely,

$$N^\mu_{;\mu} = 0 \quad (2)$$

and

$$T^{\mu\nu}_{;\nu} = 0 \quad (3)$$

where

$$N^\mu = nu^\mu \quad (4)$$

is the particle four-flow and

$$T^\mu_\nu = \frac{n\varepsilon}{c^2}u^\mu u_\nu + ph^\mu_\nu + \pi^\mu_\nu + \frac{1}{c^2}q^\mu u_\nu + \frac{1}{c^2}u^\mu q_\nu \quad (5)$$

is the stress-energy tensor. Here ε is the energy density per particle, c the speed of light and p the hydrostatic pressure. The dissipative fluxes are the Navier tensor $\pi^{\mu\nu}$ and the heat flux q^μ . We have also used the standard expression for the spatial projector

$$h^{\mu\nu} = g^{\mu\nu} + \frac{u^\mu u^\nu}{c^2}$$

in a $+++ -$ metric and the orthogonality relations

$$u^\nu q_\nu = 0 \quad u^\nu \pi_{\mu\nu} = 0$$

are also assumed. Thus, the transport equations for the relativistic fluid may be written as

$$\dot{n} + n\theta = 0 \quad (6)$$

$$\begin{aligned} \left(\frac{n\varepsilon}{c^2} + \frac{p}{c^2}\right) \dot{u}_\nu + \left(\frac{n\dot{\varepsilon}}{c^2} + \frac{p}{c^2}\theta\right) u_\nu + p_{;\mu} h_\nu^\mu + \pi_{\nu;\mu}^\mu \\ + \frac{1}{c^2} \left(q_{;\mu}^\mu u_\nu + q^\mu u_{\nu;\mu} + \theta q_\nu + u^\mu q_{\nu;\mu}\right) = 0 \end{aligned} \quad (7)$$

$$nC_n \dot{T} + p\theta + u_{;\mu}^\nu \pi_\nu^\mu + q_{;\mu}^\mu + \frac{1}{c^2} \dot{u}^\nu q_\nu = 0 \quad (8)$$

where the relation $\dot{\varepsilon} = C_n \dot{T}$ has been used in order to turn the internal energy balance equation in the heat equation. Also, we have defined $\theta = u_{;\nu}^\nu$ and a dot denotes a proper time derivative. In order to complete the set of equations, constitutive relations for the dissipative terms have to be provided. In Ref. [1] the constitutive equations proposed by Eckart, namely

$$\pi_\nu^\mu = -2\eta h_\alpha^\mu h_\nu^\beta \tau_\beta^\alpha - \zeta \theta \delta_\nu^\mu \quad (9)$$

and

$$q^\nu = -\kappa h_\mu^\nu \left(T^{;\mu} + \frac{T}{c^2} \dot{u}^\mu\right) \quad (10)$$

where introduced. In the present work we instead use the constitutive relation obtained from kinetic theory by solving Boltzmann's equation by the Chapman-Enskog approximation, that is

$$q^\nu = -h_\mu^\nu \left(L_{TT} \frac{T^{;\mu}}{T} + L_{nT} \frac{n^{;\mu}}{n}\right) \quad (11)$$

for the heat flux, while the equation for the momentum flux identical [6].

In order to account for self-gravitational effects prior to collapse, we consider a Newtonian approximation of general relativity given by the line element

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - \left(1 - \frac{2\psi(r)}{c^2}\right) (cdt)^2 \quad (12)$$

where ψ is the ordinary gravitational potential. Since, to first order in fluctuations, the continuity and heat equations, Eqs. (6) and (8), are only coupled with the velocity through its

divergence $\delta\theta$, it is convenient to calculate the divergence and curl of the momentum balance equation Eq. (7). It can be shown that by performing such calculation, the equation for the curl of the velocity decouples from the system and has decaying solutions for the corresponding fluctuations [7]. On the other hand, the equation for $\delta\theta$ now has an inhomogeneous term that can be expressed in terms of density fluctuations by using the Poisson equation. The resulting linearized system of equations reads

$$\frac{\partial}{\partial t}(\delta n) + n_0 \delta\theta = 0 \quad (13)$$

$$\tilde{\rho}_0 \frac{\partial}{\partial t}(\delta\theta) + kT_0 \nabla^2(\delta n) + n_0 k \nabla^2(\delta T) - A \nabla^2(\delta\theta) - L_{TT} \nabla^2(\delta\dot{T}) - \frac{L_{nT}}{c^2} \nabla^2(\delta\dot{n}) = -4\pi G m \tilde{\rho}_0(\delta n) \quad (14)$$

$$C_n n_0 \frac{\partial}{\partial t}(\delta T) + n_0 k T_0(\delta\theta) - L_{TT} \nabla^2(\delta T) - L_{nT} \nabla^2(\delta n) = 0 \quad (15)$$

where, for convenience we have defined $\tilde{\rho}_0 = (n_0 \varepsilon_0 + p_0)/c^2$ and considered a fluid at rest, or an observer in a comoving frame, such that the equilibrium value of the hydrodynamic velocity vanishes. Equations (13-15) constitute the system of linearized equations for the relativistic fluid to first order in the gradients considering a weak (Newtonian) gravitational field. In the next section, the set will be analyzed in Laplace-Fourier space in order to address the behavior of density fluctuations.

III. DISPERSION RELATION

In the previous section, the system of transport equations to first order, both in gradients and in fluctuations has been established. In order to analyze the behavior of density fluctuations, we proceed by following the standard method in fluctuation theory [9]. The set is transformed to Laplace-Fourier space and the dispersion relation is obtained by setting the corresponding determinant equal to zero. In this case, such a relation is given by

$$\det \begin{pmatrix} s & n_0 & 0 \\ -kT_0 q^2 + \frac{L_{nT}}{c^2} s q^2 + 4\pi G m \tilde{\rho}_0 & A q^2 + \tilde{\rho}_0 s & -n_0 k q^2 + \frac{L_{TT}}{c^2} s q^2 \\ L_{nT} q^2 & n_0 k T_0 & \frac{3}{2} k n_0 s + L_{TT} q^2 \end{pmatrix} = 0 \quad (16)$$

or, as a cubic equation

$$s^3 + a_2(q) s^2 + a_1(q) s + a_0(q) = 0 \quad (17)$$

where the coefficients are given by

$$a_2(q) = \left(\frac{A}{\tilde{\rho}_0} + \frac{L_{TT}}{n_0 C_n} - \frac{k T_0 L_{TT}}{\tilde{\rho}_0 c^2 C_n} - \frac{n_0 L_{nT}}{\tilde{\rho}_0 c^2} \right) q^2 \quad (18)$$

$$a_1(q) = \frac{A L_{TT}}{\tilde{\rho}_0 c^2 C_n} q^4 + \frac{5 k T_0 n_0}{3 \tilde{\rho}_0} q^2 - 4 \pi G m n_0 \quad (19)$$

and

$$a_0(q) = \left(\frac{k T_0 L_{TT}}{\tilde{\rho}_0 C_n} - \frac{k T_0 L_{nT}}{\tilde{\rho}_0 C_n} \right) q^4 - \frac{4 \pi G m L_{TT}}{C_n} q^2 \quad (20)$$

It is worthwhile pointing out at this stage that by considering $A = L_{nT} = L_{TT} = 0$ for the non-dissipative case, one is led directly to the Jeans wave number

$$q_J^2 = \frac{4 \pi G m n_0}{C_s^2} \quad (21)$$

where

$$C_s^2 = \frac{k T_0 n_0}{\tilde{\rho}_0} \left(1 + \frac{k}{C_n} \right) \quad (22)$$

is the speed of sound in the medium, such that the only relativistic corrections to the Jeans instability criterion neglecting dissipation arise from the relativistic values of $\tilde{\rho}_0$ and C_n . Indeed, as is verifiable from the system of equations, the relativistic effects are mostly coupled with the dissipative ones. Since in the non-relativistic limit $\tilde{\rho}_0 \sim m n_0$ we recover the usual Jeans wave number, where $C_s^2 = \frac{5 k T}{3 m}$.

Turning back to the complete dispersion relation, one can extract information about the roots of such equation by using the results obtained in the case without gravitational field. We recall that in such calculation, both relativistic and non-relativistic dispersion relations have one real solution, which gives rise to a central peak called the Rayleigh peak, and two imaginary roots which lead to damped acoustic modes corresponding to the Brillouin peaks. For a more detailed analysis of these solutions and the corresponding scattering spectrum the reader may consult Refs. [8], [9] and [10]. Based on those results, we here assume that the Rayleigh peak is not significantly modified by the presence of the gravitational field such that one of the roots of Eq. (17) still has the form $s = \gamma q^2$ and find that this condition is satisfied by the coefficient

$$\gamma = - \frac{L_{TT}}{n_0 C_n} \quad (23)$$

Under this assumption, we can factorize Eq. (17) as follows

$$(s + \gamma q^2) (s^2 + \mu s + \nu) = 0 \quad (24)$$

and, keeping only up to second order in q we obtain

$$\mu = \frac{1}{\tilde{\rho}_0} \left(A - \frac{L_{nT}}{c^2} - \frac{kT_0 L_{TT}}{c^2 C_n} \right) q^2 \quad (25)$$

and

$$\nu = -4\pi G m n_0 + \frac{kT_0 n_0}{\tilde{\rho}_0} \left(1 + \frac{k}{C_n} \right) q^2 \quad (26)$$

The factorization in Eq. (24) corresponds to the original dispersion relation given in Eq. (17) up to order q^2 which is consistent with the approximation here considered. Having two roots of the equation, arising from a second order polynomial yields the same physics as in the Jeans instability with and without dissipation, since one can find limiting values for q for which either oscillating or growing modes exist.

The limiting value for growing modes in density fluctuations, which physically leads to a gravitational collapse, is extracted directly from the discriminant of the second order polynomial in Eq. (17), that is

$$\left(A - \frac{L_{nT}}{c^2} - \frac{kT_0 L_{TT}}{c^2 C_n} \right)^2 q^4 - kT_0 n_0 \tilde{\rho}_0 \left(1 + \frac{k}{C_n} \right) q^2 + 4\pi G m n_0 \tilde{\rho}_0^2 = 0 \quad (27)$$

which has a root given by

$$q^2 = \frac{2C_s^2 \tilde{\rho}_0^2}{A \left(A - \frac{2n_0 L_{nT}}{c^2} - \frac{4T_0 L_{TT}}{3c^2} \right)} \left[1 - \sqrt{1 - \frac{4\pi G}{C_s^4 \tilde{\rho}_0} A \left(A - \frac{2n_0 L_{nT}}{c^2} - \frac{4T_0 L_{TT}}{3c^2} \right)} \right] \quad (28)$$

As mentioned before, this wave number corresponds to the limit between oscillating and exponentially growing modes. At this point it is convenient to point out that in previous work [1], the dispersion relation obtained was a fourth order polynomial with a dominating positive root which wiped out any possibility of structure formation in the system. On the other hand, the use of Eq. (11) leads to the same behavior as in the non-relativistic case, with small relativistic corrections as will be shown in the next section.

IV. CORRECTION TO THE JEANS WAVE NUMBER AND NON-RELATIVISTIC LIMIT

In order to assess the magnitude of the corrections to the Jeans wave number implied by Eq. (28) we expand the terms in the square root, so that

$$q^2 = \tilde{q}_J^2 \left[1 - \frac{\pi G A}{C_s^4 \tilde{\rho}_0} \left(A - \frac{2n_0 L_{nT}}{c^2} - \frac{4T_0 L_{TT}}{3c^2} \right) \right] \quad (29)$$

where

$$\tilde{q}_J^2 = \frac{4\pi G \tilde{\rho}_0}{C_s^2} \quad (30)$$

is the Jeans wave number corrected to the relativistic values of $\tilde{\rho}_0$ and C_s^2 . Notice that the corrections beyond $q^2 \sim \tilde{q}_J^2$ arise solely from the coupling of the gravitational effects with the dissipative ones. Moreover, the relativistic corrections depend of the heat flux but are also coupled with the viscous effect. This is an interesting result which gives further insight in the role of viscous dissipation in relativistic systems and reassures the need of more research in this direction.

The non-relativistic limit is readily obtained when $c \rightarrow \infty$ since $q^2 \rightarrow \tilde{q}_J^2$ and we obtain

$$q^2 = \tilde{q}_J^2 \left(1 - \frac{\pi G A^2}{C_s^4 \tilde{\rho}_0} \right) \quad (31)$$

which is precisely the result in Ref. [11] with $q^2 = \tilde{q}_J^2$ included in Eq. (31), for the case of a non-dissipative medium. Thus, the introduction of a constitutive equation for the heat flux in terms of gradients of scalar state variables - temperature, density, pressure - assures that the gravitational collapse is possible for relativistic systems following roughly the same physics as in the non-relativistic scenario, that is with the existence of a critical wavenumber indicating a range of values for which a structure will form within the gas.

V. CONCLUSIONS AND FINAL REMARKS

The issue of which is the correct formalism for describing dissipative processes in relativistic hydrodynamics has recently been a subject of intense research. It was already shown that the causality and stability issues allegedly present in first order theories are not a concern when the constitutive equations arising from the microscopic formalism are considered [7, 12, 13]. The causality of the system is guaranteed as long as one considers fluctuations in the hydrodynamic velocity even in a comoving frame [14]. This is a completely reasonable assumption since either a fluid at rest or an observer in a comoving frame shall be stated as a vanishing *equilibrium* velocity. Fluctuations, which are physical evidence of the statistical picture underlying the hydrodynamic formalism, are spontaneous and always present. Thus, there is really no reason to consider $\delta\theta = 0$. On the other hand, the stability concern is actually more a problem about the relaxation of such fluctuations instead of a hydrodynamic

instability *per-se*. In any case, it has been shown that the pathological behavior of the system is wiped out when gradients are included in the momentum balance through the heat flux constitutive equation instead of the acceleration which appears in Eckart's formalism [4]. These facts lead to the firm statement that extended theories are not *strictly necessary* and first order theories are physically sound as long as the results of kinetic theory are used to close the system of transport equations. However, as shown in Ref. [12], introducing relaxation parameters also solves the problem by forcing into the equations a decay that overrides the instability.

The question of which formalism is more adequate is still a subject of debate. In the present paper the calculation presented in Ref. [1] by two of us is carried out once again, but now using the closure relation obtained from kinetic theory in order to argue in favor of first order theories. It is a fact that structures form in systems when the Jeans criterion is met. It is desirable to have also this kind of behavior in the relativistic formalism with corrections which vanish for mild or low temperatures. In Ref. [1] it was clearly shown that if Eckart's closure is introduced in the transport equations, the gravitational instability is not present in the system, even in the non-relativistic limit. On the other hand, what has been shown here is that by introducing the constitutive equation obtained in Refs. [5–7], the physics in the gravitational collapse in a relativistic system matches the phenomenon established by Jeans in the non-dissipative case [15] as well as the one described by Sandoval-Villalbazo and García-Colín for dissipative systems [11]. The mechanism for the onset of the gravitational instability is the same and only slight corrections to the Jeans wave number are obtained. It is of the authors' opinion that even if these corrections are not astrophysically significant, the fact that the theory predicts a collapse under particular conditions and that it yields the correct non-relativistic limit is enough evidence to rule it as more adequate to describe the physics of relativistic gases than other proposals.

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