# Angular Spectra for non-Gaussian Isotropic Fields

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#### Abstract

Cosmic Microwave Background (CMB) Anisotropies is a subject of intensive research in several fields of sciences [HD02]. In this paper we start a systematic development of basic notions and theory in statistics according to the application for CMB. The main result of this paper is the necessary and sufficient condition for isotropy of a non-Gaussian field in terms of spectra. Clear formulae for bi-, tri- and polyspectra and bi-, tri-, and higher order covariances are also given.

Keywords: Bispectrum, Trispectrum, Angular poly-Spectra, Cosmic microwave background radiation; Gaussianity; spherical random fields

## 1 Introduction

"Cosmological Principle (first coined by Einstein): the Universe is, in the large, homogeneous and isotropic" [Bar99]

In the last decade or so there has been some growing interest in studying the spatial - time data measured on the surface of a sphere. These data includes cosmic microwave background (CMB) anisotropies [KK06], [OH02], [AH12], medical imaging [KBID93] [Kak12], global and land-based temperature data [Jon94], [SRT06], gravitational and geomagnetic data, climate model [NC81].

One of the problem in focus is the Non-Gaussianity of the observations, which leads to the investigation of higher order angular spectra called polyspectra, [Hu01], [BSMR10], [BLSH12]. Angular polyspectra, in particular the bispectrum and trispectrum shows to be an appropriate measure of Gaussianity since for a Gaussian process all higher (then second order) spectra are zero, [KSH11]. An other important question is the Monte Carlo simulation of non-Gaussian isotropic maps with a given power spectrum and bispectrum and possible polyspectrum, [CM01], [RHS+05].

In this paper some general properties of the angular polyspectra will be given for isotropic stochastic processes on sphere. Our treatment follows the basic theory of higher order spectra

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for non-Gaussian time series, [Bri65], [Bri01], [SRG84], [Ter99]. The symmetry relations and the appropriate series expansion of cumulants might have influence on the estimation of polyspectra as well, [SK12], [IMK12].

The isotropy assumption implies very particular form for the angular spectra therefore a delicate question is the construction of isotropic stochastic maps on the sphere.

# 2 Gaussian isotropic fields

Gaussian isotropic processes on sphere has a long history starting with [Obu47], some basic theory and references can be found in [Yag61], [Jon63], [McL86] [Yad83] [Yag87], ([Leo99]). Due to the expansive recent applications there are new books [GG10], [CW11] and several papers covering a number of problems in general for spatial-time observations.

We consider a stochastic process X(L) on the unit sphere  $\mathbb{S}_2$  in  $\mathbb{R}^3$ , where  $L = (\vartheta, \varphi)$ , the co-latitude  $\vartheta \in [0, \pi]$  and the longitude  $\varphi \in [0, 2\pi]$ . Let us suppose that X(L) is continuous (in mean square sense), then it has a series expansion in terms of spherical harmonics  $Y_{\ell}^m$ 

$$X(L) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell}^{m} Y_{\ell}^{m} (L), \qquad (1)$$

where the coefficients are given by

$$Z_{\ell}^{m} = \int_{\mathbb{S}_{2}} X\left(L\right) \overline{Y_{\ell}^{m}\left(L\right)} \Omega\left(dL\right),$$

where  $\Omega\left(dL\right)=\sin\vartheta d\vartheta d\varphi$  is Lebesque element of surface area on  $\mathbb{S}_2$ . It is generally accepted, in time series analysis that  $X\left(L\right)$  called linear if  $Z_{\ell}^{m}$  are independent and identically distributed ( $\mathsf{E}X\left(L\right)=0$ ). Now it is straightforward that from the linearity here follows that the covariance function  $\mathcal{C}_2\left(L_1,L_2\right)=\mathsf{E}X\left(L_1\right)X\left(L_2\right)$  depends on the angular distance  $\gamma$  between  $L_1$  and  $L_2$  only. Necessarily the covariance function depends on the central angle between locations and has the form

$$C_2(L_1, L_2) = C(\cos \gamma) = \sum_{\ell=0}^{\infty} f_{\ell} \frac{2\ell+1}{4\pi} P_{\ell}(\cos \gamma), \qquad (2)$$

where  $P_{\ell}$  denotes the Legendre polynomial, see [Yad83], I.5., and all coefficients  $f_{\ell} \geq 0$ . Note here that the inner product  $L_1 \cdot L_2 = \cos \gamma$ . In other words  $C_2(L_1, L_2)$  is invariant under the group of rotations, i.e. X(L) is isotropic. Moreover the assumption of linearity, i.e. the independence of the triangular array  $Z_{\ell}^m$  is so strong that the Gaussianity also follows, [BM07]. In other words the only linear field on the sphere is the Gaussian one. We shall consider a linearity which looks weaker although it will be shown to be equivalent to the whole independence of  $Z_{\ell}^m$ .

**Definition 1** The field X(L) is linear if the generating array  $Z_{\ell}^m$  is uncorrelated and for fixed degree  $\ell$ ,  $\{Z_{\ell}^m | m = -\ell, -\ell+1, \ell, \ell-1, \ell\}$  are independent.

This concept of linearity corresponds to the physical approach to the theory of angular momentum. One may consider linearity in some really weaker sense, namely the rows  $\{Z_\ell^m\}_{m=-\ell}^\ell$  of the array  $Z_\ell^m$ , are independent but the variables  $Z_\ell^m$  inside the row  $\ell$  are uncorrelated.

The convergence of the series  $\sum_{\ell=0}^{\infty} f_{\ell} \frac{2\ell+1}{4\pi}$  is equivalent to the continuity of  $\mathcal{C}(\cdot)$  on [-1,1]. The superposition (2) corresponds to the superposition of a covariance function on the real line in terms of its spectrum and the orthogonal system  $\{\exp(i2\pi\lambda k), k=0,1,2\ldots\}$ , hence we treat  $\ell$  as the frequency and  $\widehat{\mathcal{C}}_{\ell} = f_{\ell}$  is the value of the spectrum at  $\ell$ . Since  $P_{\ell}(\cos 0) = 1$ , the variance  $\mathsf{E}X(L)^2$  is broken up into the sum of spectra therefore we have the analysis of variance. The inversion

$$\widehat{C}_{\ell} = \int_{\mathbb{S}_2} C(\cos \gamma) P_{\ell}(\cos \gamma) \Omega(dL),$$

is also valid. The stochastic process X(L) itself has spectral representation (1) also in terms of spherical harmonics  $Y_{\ell}^{m}$  with complex values, detailed account for spherical harmonics  $Y_{\ell}^{m}$  is found in [VMK88], [SW71]. For a given X(L) we have the inversion

$$Z_{\ell}^{m} = \int_{\mathbb{S}_{2}} X\left(L\right) \overline{Y_{\ell}^{m}\left(L\right)} \Omega\left(dL\right).$$

The orthogonal random 'measure'  $Z_{\ell}^m$  is a triangular array for each fix  $\ell$ ;  $m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell$ , i.e. rows contain  $2\ell + 1$ , i.i.d Gaussian random variables,  $\mathsf{E}Z_{\ell}^m = 0$ ,  $\mathsf{E}Z_{\ell}^m Z_k^{n*} = f_{\ell}\delta_{\ell,k}\delta_{m,n}$ . The reason that  $f_{\ell}$  does not depend on m, is the Funk-Hecke formula (26), see [Yad83], I.5., [Jon63].

The  $C_2(L_1 \cdot L_2)$  is strictly positive definite if all  $f_\ell$  is  $\geq 0$ , and only finitely many are zero ([Sch97], [Sch42]).

**Example 2** Matérn Class of Covariance Functions [Mat86], [SG05] Suppose one has a covariance function on the real line then its restriction to  $[0,\pi]$  gives a covariance function  $C_0(\gamma)$ , on the sphere  $S_2$ ,  $C_0(\gamma) = C(\cos \gamma)$ , for instance

$$C_{0}(\gamma) = \sigma^{2} \frac{1}{\Gamma(\nu)} \left(\frac{\vartheta \gamma}{2}\right)^{\nu} 2K_{\nu}(\vartheta \gamma), \ \vartheta > 0, \ \nu > 0,$$
$$f_{\ell} = \int_{0}^{\pi} C_{0}(\gamma) \frac{2\ell + 1}{4\pi} P_{\ell}(\cos \gamma) \, d\gamma.$$

Where  $K_{\nu}$  is the modified Bessel function of the second kind. Here the smoothness parameter  $\nu$  controls the continuity, as well as  $\vartheta$  controls the regularity [GG10], [She11]. Note  $K_{\nu}(r) \sim \Gamma(\nu) (r/2)^{-\nu}/2$  if  $r \to 0$ , and  $\nu > 0$ , hence  $C_0(\gamma)/\sigma^2$  is a correlation function.

**Example 3** The generating function of Legendre polynomial  $P_{\ell}$  is

$$\sum_{\ell=0}^{\infty} P_{\ell}(x) z^{\ell} = \left(1 - 2xz + z^{2}\right)^{-1/2}, \ x \in (-1,1), \ |z| < 1$$

see (24). Let 0 < z < 1, hence

$$C(\cos \gamma) = \frac{1}{\sqrt{1 - 2z\cos \gamma + z^2}},$$

is a covariance function on  $\cos \gamma \in [-1,1]$ , with  $f_{\ell} = \frac{4\pi}{2\ell+1}z^{\ell}$ . Similarly for any n > 2 we have a covariance function

$$C(\cos \gamma) = \frac{1}{(1 - 2z\cos \gamma + z^2)^{(n-2)/2}},$$

if 0 < z < 1.

Example 4 From the well known series

$$\sum_{\ell=0}^{\infty} P_{\ell}(x) \frac{a^{\ell}}{2\ell+1} = \frac{1-a^2}{(1-2xa+a^2)^{3/2}}, \ x \in (-1,1), \ |a| < 1$$

see [Yad83], for any 0 < a < 1, we find the covariance function

$$C(\cos \gamma) = \frac{1 - a^2}{(1 - 2a\cos \gamma + a^2)^{3/2}},$$

on [-1,1] and  $f_{\ell} = 4\pi a^{\ell}$ . Similarly for  $n \geq 2$ , we have

$$C(\cos \gamma) = \frac{1 - a^2}{(1 - 2a\cos \gamma + a^2)^{n/2}}.$$

Moreover the series

$$\sum_{\ell=1}^{\infty} P_{\ell}(x) \frac{t^{\ell}}{\ell} = \ln \frac{2}{1 - tx + \sqrt{1 - 2xt + t^2}}, \quad 0 < t < 1, \ |x| < 1$$

see [PBM86], 5.10.1.4., gives a covariance function as well.

**Example 5** It is known, [Bre63], [WC01], that the probability density on the sphere

$$C(\cos \gamma) = \frac{\kappa}{4\pi \sinh(\kappa)} \exp(\kappa \cos \gamma), \ \kappa > 0$$

has series expansion

$$C\left(\cos\gamma\right) = \frac{\kappa}{\sinh\left(\kappa\right)} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{\frac{\pi}{2\kappa}} I_{\ell+1/2}\left(\kappa\right) P_{\ell}\left(\cos\gamma\right),\,$$

where  $\sqrt{\frac{\pi}{2\kappa}}I_{\ell+1/2}(\kappa)$  is the modified spherical Bessel function of the first kind, [AS92], also

$$(2\ell+1)\sqrt{\frac{\pi}{2\kappa}}I_{\ell+1/2}(\kappa)\sim \frac{\kappa^{\ell}}{(2\ell-1)!!}.$$

Hence C is a covariance function with spectrum

$$f_{\ell} = \frac{\kappa}{\sinh(\kappa)} \sqrt{\frac{\pi}{2\kappa}} I_{\ell+1/2}(\kappa)$$
$$= \frac{I_{\ell+1/2}(\kappa)}{I_{1/2}(\kappa)}.$$

Note  $\sinh\left(\kappa\right)/\kappa = \sqrt{\frac{\pi}{2\kappa}}I_{1/2}\left(\kappa\right)$ , and

$$\exp(a\cos\vartheta) = \sum_{\ell=0}^{\infty} (2\ell+1) \sqrt{\frac{\pi}{2a}} I_{\ell+1/2}(a) P_{\ell}(\cos\vartheta)$$

see [AS92], [PBM86].

**Example 6** For  $\kappa > 0$ ,  $\gamma \in [0, \pi]$  we have

$$\sum_{\ell=0}^{\infty} \frac{\kappa^{\ell}}{\ell!} P_{\ell} (\cos \gamma) = \exp (\kappa \cos \gamma) J_0 (\kappa \sin \gamma),$$

$$f_{\ell} = \frac{\kappa^{\ell}}{\ell!} \frac{4\pi}{2\ell + 1},$$

see [PBM86], hence  $\exp(\kappa \cos \gamma) J_0(\kappa \sin \gamma) / \exp(\kappa)$  is a correlation function, where  $J_0$  denotes Bessel function of the first kind.

Example 7 We have a covariance function of the form

$$C(\cos \gamma) = I_0 \left( 2\kappa \sqrt{\cos \gamma - 1} \right) I_0 \left( 2\kappa \sqrt{\cos \gamma + 1} \right)$$
$$= \sum_{\ell=0}^{\infty} \frac{\kappa^{\ell}}{(\ell!)^2} P_{\ell} (\cos \gamma) ,$$
$$f_{\ell} = \frac{\kappa^{\ell}}{(\ell!)^2} \frac{4\pi}{2\ell + 1}$$

see [PBM86]. Note that  $I_0(x)$  is a function of  $x^2/4$  see [AS92], hence there is no danger of complex values.  $I_0(0) = 1$ ,  $I_0$  denotes modified Bessel function of the first kind.

**Example 8** Poisson formula. For a homogeneous isotropic field on  $\mathbb{R}^3$  we have the spectral representation

$$C(r) = \int_{0}^{\infty} j_{0}(\lambda r) \Phi(d\lambda),$$

of a covariance function with spectral measure  $\Phi(d\lambda)$ , see [Yad83], where  $j_0$  is the Spherical Bessel function of the first kind, see [AS92]. If we consider two locations  $L_1$  and  $L_2$  on the sphere  $\mathbb{S}_2$  with angle  $\gamma \in [0, \pi]$ , then the distance  $r = ||L_1 - L_2||$  between them in term of the angle is  $2\sin(\gamma/2)$ , and  $L_1 \cdot L_2 = \cos\gamma$ . Hence we have the covariance function on sphere  $\mathcal{C}_0(\cos\gamma) = \mathcal{C}(2\sin(\gamma/2))$ .  $\mathcal{C}_0(\cos\gamma)$  defines an isotropic field on the sphere  $\mathbb{S}_2$  with spectrum

$$f_{\ell} = 2\pi^2 \int_0^{\infty} J_{\ell+1/2}^2(\lambda) \frac{1}{\lambda} \Phi(d\lambda), \qquad (3)$$

where  $J_{\ell+1/2}$  denotes the Bessel function of the first kind, see [AS92].

**Example 9** Lapalce-Beltrami model on  $\mathbb{S}_2$ . Consider the homogeneous isotropic field X on  $\mathbb{R}^3$  according to the equation

$$(\triangle - c^2) X = \partial W,$$

where  $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ , denotes the Laplace operator on  $\mathbb{R}^3$ . Its spectrum, see ([Yad83]), is

$$S(\lambda) = \frac{2}{(2\pi)^2} \frac{\lambda^2}{(\lambda^2 + c^2)^2}, \quad \lambda^2 = \|(\lambda_1, \lambda_2, \lambda_3)\|^2,$$

with covariance of Matérn Class

$$C(r) = \frac{1}{(2\pi)^{3/2}} \frac{(cr)^{1/2} K_{1/2}(cr)}{2c},$$

where  $K_{1/2}$  is the modified Bessel (Hankel) function, see [AS92]. Now the according to the Lapalce-Beltrami operator

$$\Delta_B = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2},$$

we consider the stochastic model

$$\left(\triangle_B - c^2\right) X_B = \partial W_B,$$

on sphere. The covariance function  $C_0$  of  $X_B$  is the restriction of the covariance function C of X on sphere and  $C_0(\cos \gamma) = C(2\sin (\gamma/2))$ , i.e.

$$C_0\left(\cos\gamma\right) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{\sin\left(\gamma/2\right)}{2c}} K_{1/2}\left(2c\sin\left(\gamma/2\right)\right).$$

We apply the Poisson formula (3) when  $\Phi(d\lambda) = S(\lambda) d\lambda$ , and we obtain the spectrum for  $X_B$ 

$$f_{\ell} = 2\pi^{2} \int_{0}^{\infty} J_{\ell+1/2}^{2}(\lambda) \frac{1}{\lambda} \frac{2}{(2\pi)^{2}} \frac{\lambda^{2}}{(\lambda^{2} + c^{2})^{2}} d\lambda$$
$$= \int_{0}^{\infty} J_{\ell+1/2}^{2}(\lambda) \frac{\lambda}{(\lambda^{2} + c^{2})^{2}} d\lambda,$$
$$= \frac{1}{(\ell(\ell+1) + c^{2})^{2}}.$$

## 3 Non-Gaussian isotropy and the angular spectrum

In a physical phenomenon the isotropic property is treated as a principle. It means that there is no reason to make difference between directions. The corresponding property of a stochastic field is that the finite dimensional distributions remain unchanged after rotating the space.

## 3.1 Isotropy on sphere

**Definition 10** A stochastic field X(L) on the unit sphere  $\mathbb{S}_2$  is isotropic (in strict sense) if all finite dimensional distributions of  $\{X(L), L \in \mathbb{S}_2\}$  are invariant under the rotation g for every  $g \in SO(3)$ .

From now on we do not assume Gaussianity. But, for the simplicity, we suppose the existence of moments and that those determine the distribution as well. The general form of a mean square continuous field is given in terms of spherical harmonics

$$X(L) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell}^{m} Y_{\ell}^{m} (L), \qquad (4)$$

where  $\{Z_\ell^m; \ell=0,1,\ldots;\ m=-\ell,1-\ell,\ldots,-1,0,1,\ldots,\ell-1,\ell\}$  is a triangular array, in the row  $\ell$  we have  $2\ell+1$  elements. Notice that  $Y_0^0=1$ , put  $Z_0^0=\mu$ , otherwise  $\mathsf{E}Z_\ell^m=0$ , therefore  $\mathsf{E}X\ (L)=\mu$ , and the convergence is valid in mean square sense [Leo99]. This development follows also from the stochastic Peter-Weyl Theorem, see [MP11] for details.

In case the  $m^{th}$  order cumulants of X are invariant under the rotation g for every  $g \in SO(3)$  then it will be called *isotropic in*  $m^{th}$  order. Naturally a strictly isotropic field with  $m^{th}$  order moments is isotropic in  $m^{th}$  order.

We list here some properties which follows from isotropy, see [BM07] for details,

- 1. The distribution of both Re  $Z_{\ell}^{m} \stackrel{d}{=} \operatorname{Im} Z_{\ell}^{m}$  and Re  $Z_{\ell}^{m} / \operatorname{Im} Z_{\ell}^{m}$  are Cauchy.
- 2. Marginal distribution of both Re  $Z_{\ell}^{m}$  and Im  $Z_{\ell}^{m}$  are always symmetric
- 3. For fixed  $\ell$ ,  $Z_{\ell}^{m}$ ,  $m=0,1,\ldots,\ell-1,\ell$  are independent iff they are Gaussian.

Now, let us consider a rotation  $g \in SO(3)$ , it is known that the spherical harmonics  $Y_{\ell}^{m}$  at the rotated location are given in terms of the Wigner D-matrix, see 18 Appendix B, more precisely

$$\Lambda\left(g\right)Y_{\ell}^{m}\left(L\right) = \sum_{k=-\ell}^{\ell} D_{k,m}^{\left(\ell\right)}\left(g\right)Y_{\ell}^{k}\left(L\right)$$

where  $\Lambda(g)$  denotes the operator according to the rotation g,  $\Lambda(g)Y_{\ell}^{k}(L) = Y_{\ell}^{k}(g^{-1}L)$ . Hence the rotated field has the following form

$$\begin{split} \Lambda\left(g\right)X\left(L\right) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell}^{m} \sum_{k=-\ell}^{\ell} D_{k,m}^{(\ell)}\left(g\right) Y_{\ell}^{k}\left(L\right) \\ &= \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} \sum_{m=-\ell}^{\ell} D_{k,m}^{(\ell)}\left(g\right) Z_{\ell}^{m} Y_{\ell}^{k}\left(L\right). \end{split}$$

The isotropy assumption is equivalent to that the distribution of the variable

$$\mathcal{Z}_{\ell}^{k}\left(g\right) = \sum_{m=-\ell}^{\ell} D_{k,m}^{(\ell)}\left(g\right) Z_{\ell}^{m},\tag{5}$$

is the same as the one of  $\mathbb{Z}_{\ell}^k$ . This statement will be used frequently below.

In matrix form

$$\mathcal{Z}_{\ell}(g) = D^{(\ell)}(g) Z_{\ell}$$

where  $Z_{\ell} = \left(Z_{\ell}^{-\ell}, Z_{\ell}^{-\ell+1}, \dots, Z_{\ell}^{\ell}\right)^{\top}$ ,  $D^{(\ell)} = \left(D_{k,m}^{(\ell)}\right)_{-\ell,-\ell}^{\ell,\ell}$  (the dependence on g will be omitted unless it is necessary). This equation provides an equation for the cumulant functions  $\Phi_{Z}^{\ell}$  (log of the characteristic function) as well, in case of isotropy, for each rotation g we have

$$\Phi_{Z}^{\ell}\left(\underline{\omega}_{\ell}\right) = \Phi_{Z}^{\ell}\left(\underline{\omega}_{\ell}D^{(\ell)}\left(g\right)\right),$$

where  $\underline{\omega}_{\ell} = (\omega_{-\ell}, \omega_{-\ell+1}, \dots, \omega_{\ell-1}, \omega_{\ell})$ . Hence the distribution of  $Z_{\ell}$  should be rotationally invariant on  $\mathbb{R}^{2\ell+1}$ . For instance the covariance matrix  $\mathcal{C}_{Z}(\ell_{1}, \ell_{2}) = \operatorname{Cov}(Z_{\ell_{1}}, Z_{\ell_{2}})$  commutes with  $D^{(\ell_{n})}$ , since  $D^{(\ell_{n})}$  is unitary and  $\mathcal{C}_{Z}(\ell_{1}, \ell_{2}) = D^{(\ell_{1})*}\mathcal{C}_{Z}(\ell_{1}, \ell_{2}) D^{(\ell_{2})}$ . If  $\ell_{1} = \ell_{2} = \ell$ , the only matrix which commutes with  $D^{(\ell)}(g)$ , for any  $g \in SO(3)$  is constant times unit matrix.

Assume for a moment that  $\operatorname{Cum}_2\left(Z_{\ell_1}^k,Z_{\ell_2}^m\right)$  does depend on  $\ell_1,k,\ell_2,m$ , then we show below that from the isotropy  $\operatorname{Cum}_2\left(Z_{\ell_1}^k,Z_{\ell_2}^{m*}\right)=\delta_{\ell_1,\ell_2}\delta_{k,m}f_{\ell_1}$ , follows. Hence we should restrict ourselves on an uncorrelated generating array  $Z_\ell^m$ , with  $\operatorname{E} Z_\ell^m Z_k^{n*}=\delta_{\ell,k}\delta_{m,n}\sigma_{\ell,m}^2$ . Since  $\sigma_{\ell,m}^2$  does not depend on m we denote it  $f_\ell$ , moreover we shall consider the field X in the following form

$$X\left(L\right) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell}^{m} Y_{\ell}^{m}\left(L\right),$$

where  $\mathsf{E} Z_\ell^m Z_k^{n*} = \delta_{\ell,k} \delta_{m,n} f_\ell$ . In other words it is seen that the second and higher order structure of the generating process  $Z_\ell^m$  inside the same degree  $\ell$  are hiding, they are not identifiable.

According to the angular momentum of degree  $\ell$  we define the field

$$u_{\ell}(L) = \sum_{m=-\ell}^{\ell} Z_{\ell}^{m} Y_{\ell}^{m}(L).$$

$$(6)$$

We shall be interested in the invariance of the distribution of  $u_{\ell}(L)$  under rotations as well. If the location is fixed then  $u_{\ell}(L)$  is connected to the distribution of  $\mathcal{Z}_{\ell}^{k}$  directly, since the Condon and Shortley phase convention  $\sqrt{\frac{2\ell+1}{4\pi}}D_{m,0}^{(\ell)}(g) = Y_{\ell}^{m*}(\vartheta,\varphi)$ , where the rotation g given in Euler coordinates  $(\gamma,\vartheta,\varphi)$ , provides

$$u_{\ell}(L) = \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m=-\ell}^{\ell} D_{0,m}^{(\ell)*}(\gamma, \vartheta, \varphi) Z_{\ell}^{m},$$

where the notation \* defined as the transpose and conjugate for a matrix and just conjugate for a scalar. Observe that  $\gamma$  here is arbitrary. Our main interest is the comparison of the finite dimensional distributions of the field  $u_{\ell}(L)$  to the rotated one. In case a rotation carries the location L to the North pole N,  $u_{\ell}$  simplifies

$$u_{\ell}\left(N\right) = \sqrt{\frac{2\ell+1}{4\pi}} Z_{\ell}^{0}.$$

Rewrite X into the form

$$X(L) = \sum_{\ell=0}^{\infty} u_{\ell}(L).$$

We consider a real valued X(L), therefore  $Z_{\ell}^{m*} \stackrel{d}{=} (-1)^m Z_{\ell}^{-m}$ , since  $Y_{\ell}^m(L)^* = (-1)^m Y_{\ell}^{-m}(L)$ . Moreover if we reflect the location to the center then from  $Y_{\ell}^m(-L) = (-1)^{\ell} Y_{\ell}^m(L)$  follows

$$X(-L) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (-1)^{\ell} Z_{\ell}^{m} Y_{\ell}^{m} (L).$$
 (7)

Therefore we have the following

**Remark 11** Since the parity  $u_{\ell}(-L) = (-1)^{\ell} u_{\ell}(L)$  is valid, it implies that under assumption of isotropy

$$\operatorname{Cum}_{p}\left(u_{\ell_{1}}\left(L_{1}\right),\ldots,u_{\ell_{p}}\left(L_{p}\right)\right)=\left(-1\right)^{\ell_{1}+\ell_{2}+\cdots+\ell_{p}}\operatorname{Cum}_{p}\left(u_{\ell_{1}}\left(L_{1}\right),\ldots,u_{\ell_{p}}\left(L_{p}\right)\right),$$

hence for p > 2 the sum  $\mathcal{L}_p = \ell_1 + \ell_2 + \cdots + \ell_p$  must be even.

#### 3.1.1 Second order Isotropy

Consider the second order cumulants of  $\mathcal{Z}_{\ell}^{k}$ , defined in (5),

$$\operatorname{Cum}_{2}\left(\mathcal{Z}_{\ell_{1}}^{k}, \mathcal{Z}_{\ell_{2}}^{m}\right) = \sum_{p,q=-\ell_{1},\ell_{2}}^{\ell_{1},\ell_{2}} D_{k,p}^{(\ell_{1})} D_{m,q}^{(\ell_{2})} \operatorname{Cum}_{2}\left(Z_{\ell_{1}}^{p}, Z_{\ell_{2}}^{q}\right)$$

$$= \sum_{p,q=-\ell_{1},\ell_{2}}^{\ell_{1},\ell_{2}} D_{k,p}^{(\ell_{1})} \left(-1\right)^{p} D_{m,q}^{(\ell_{2})} \operatorname{Cum}_{2}\left(Z_{\ell_{1}}^{-p*}, Z_{\ell_{2}}^{q}\right).$$

Now assume isotropy,  $\operatorname{Cum}_2\left(\mathcal{Z}_{\ell_1}^k,\mathcal{Z}_{\ell_2}^m\right) = \operatorname{Cum}_2\left(Z_{\ell_1}^k,Z_{\ell_2}^m\right)$  and integrate both sides over the sphere according to the invariant Haar measure, we have

$$\operatorname{Cum}_{2}\left(Z_{\ell_{1}}^{k}, Z_{\ell_{2}}^{m}\right) = \sum_{p,q=-\ell_{1},\ell_{2}}^{\ell_{1},\ell_{2}} \frac{\delta_{\ell_{1},\ell_{2}}\delta_{-p,q}\delta_{-k,m}\left(-1\right)^{k}}{2\ell_{1}+1} \operatorname{Cum}_{2}\left(Z_{\ell_{1}}^{-p*}, Z_{\ell_{2}}^{q}\right)$$

$$= \frac{\delta_{-k,m}\left(-1\right)^{m}}{2\ell_{1}+1} \sum_{p=-\ell_{1}}^{\ell_{1}} \delta_{\ell_{1},\ell_{2}} \operatorname{Cum}_{2}\left(Z_{\ell_{1}}^{-p*}, Z_{\ell_{2}}^{p}\right)$$

$$= \delta_{\ell_{1},\ell_{2}}\delta_{-k,m}\left(-1\right)^{m} C_{2}\left(\ell_{1}\right)$$

where

$$C_2(\ell_1) = \frac{1}{2\ell_1 + 1} \sum_{p=-\ell_1}^{\ell_1} \text{Cum}_2(Z_{\ell_1}^{-p*}, Z_{\ell_1}^p).$$

Hence  $\operatorname{Cum}_2\left(Z_{\ell_1}^k,Z_{\ell_2}^{m*}\right)=\delta_{\ell_1,\ell_2}\delta_{k,m}f_{\ell_1}$ , i.e. the series  $Z_\ell^k$  is uncorrelated. If the series  $Z_\ell^k$  is uncorrelated then

$$\operatorname{Cum}_{2}\left(\mathcal{Z}_{\ell_{1}}^{k}, \mathcal{Z}_{\ell_{2}}^{m}\right) = \sum_{p,q=-\ell_{1},\ell_{2}}^{\ell_{1},\ell_{2}} D_{k,p}^{(\ell_{1})} D_{m,q}^{(\ell_{2})} \operatorname{Cum}_{2}\left(\mathcal{Z}_{\ell_{1}}^{p}, \mathcal{Z}_{\ell_{2}}^{q}\right)$$

$$= \delta_{\ell_{1},\ell_{2}} \sum_{p,q=-\ell_{1}}^{\ell_{1}} D_{k,p}^{(\ell_{1})} D_{m,q}^{(\ell_{1})} (-1)^{p} \operatorname{Cum}_{2}\left(\mathcal{Z}_{\ell_{1}}^{-p*}, \mathcal{Z}_{\ell_{1}}^{q}\right)$$

$$= \delta_{\ell_{1},\ell_{2}} \sum_{p=-\ell_{1}}^{\ell_{1}} D_{-p,-k}^{(\ell_{1})*} D_{m,q}^{(\ell_{1})} (-1)^{k} \delta_{-p,q} f_{\ell_{1}}$$

$$= \delta_{\ell_{1},\ell_{2}} \delta_{-k,m} (-1)^{m} f_{\ell_{1}}$$

since  $D_{k,m}^{(\ell)}$  is unitary, hence  $\operatorname{Cum}_2\left(\mathcal{Z}_{\ell_1}^k,\mathcal{Z}_{\ell_2}^{-m*}\right)=\delta_{\ell_1,\ell_2}\delta_{-k,m}f_{\ell_1}=\operatorname{Cum}_2\left(Z_{\ell_1}^k,Z_{\ell_2}^{-m*}\right)$ .

**Lemma 12** The field X(L) is isotropic in second order iff the triangular series  $Z_{\ell}^{k}$  is uncorrelated with variance  $f_{\ell}$ .

We conclude that a field  $X\left(L\right)$  with Gaussian i.i.d.  $Z_{\ell}^{m}$  is strictly isotropic. It follows

from the isotropy that  $u_{\ell_1}$  and  $u_{\ell_2}$  with different degrees  $\ell_1$  and  $\ell_2$  are uncorrelated, and

$$\operatorname{Cum}_{2}(u_{\ell}(L_{1}), u_{\ell}(L_{2})) = \sum_{m_{1}, m_{2} = -\ell}^{\ell} \operatorname{Cum}_{2}(Z_{\ell}^{m_{1}}, Z_{\ell}^{m_{2}}) Y_{\ell}^{m_{1}}(L_{1}) Y_{\ell}^{m_{2}}(L_{2})$$

$$= \sum_{m_{2} = -\ell}^{\ell} f_{\ell} Y_{\ell}^{-m_{2}}(L_{1}) Y_{\ell}^{m_{2}}(L_{2}) (-1)^{m_{2}}$$

$$= f_{\ell} \sum_{m_{2} = -\ell}^{\ell} Y_{\ell}^{m_{2}*}(L_{1}) Y_{\ell}^{m_{2}}(L_{2})$$

$$= \frac{2\ell + 1}{4\pi} f_{\ell} P_{\ell}(L_{1} \cdot L_{2}),$$

by the addition formula see (42). In particular  $\operatorname{Var}(u_{\ell}(L)) = \frac{2\ell+1}{4\pi} f_{\ell}$ . Instead of the addition formula one arrives to the same result if applies the isotropy for  $\operatorname{Cum}_2(u_{\ell}(L_1), u_{\ell}(L_2))$ . Rotate the locations  $L_1$  and  $L_2$  such that  $L_2$  becomes the North pole N and at the same time  $L_1$  belongs to the plane xOz. This rotation will be denoted by  $g_{L_2L_1}$ . We have  $\operatorname{Cum}_2(u_{\ell}(L_1), u_{\ell}(L_2)) = \operatorname{Cum}_2(u_{\ell}(g_{L_2L_1}L_1), u_{\ell}(N))$ , since  $u_{\ell}(N) = \sqrt{\frac{2\ell+1}{4\pi}} Z_{\ell}^0$  and  $Y_{\ell}^0(\vartheta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}(\cos \vartheta)$ ,

$$\operatorname{Cum}_{2}(u_{\ell}(L_{1}), u_{\ell}(L_{2})) = \operatorname{Cum}_{2}\left(u_{\ell}(g_{L_{2}L_{1}}L_{1}), \sqrt{\frac{2\ell+1}{4\pi}}Z_{\ell}^{0}\right)$$

$$= \operatorname{Cum}_{2}\left(\sqrt{\frac{2\ell+1}{4\pi}}P_{\ell}(g_{L_{2}L_{1}}L_{1}\cdot N)Z_{\ell}^{0}, \sqrt{\frac{2\ell+1}{4\pi}}Z_{\ell}^{0}\right)$$

$$= \frac{2\ell+1}{4\pi}f_{\ell}P_{\ell}(L_{1}\cdot L_{2}).$$

#### 3.2 Spectrum

Consider the covariance function  $C_2(L_1, L_2) = \mathbb{E}(X(L_1) - \mu)(X(L_2) - \mu)$  for an isotropic field. Let the rotation  $g_{L_2L_1}$  be the one which takes the location  $L_2$  into the North pole N, and  $L_1$  into the plane xOz. The Euler coordinates of  $g_{L_2L_1}L_1$  is the co-latitude angle  $\vartheta$  and 0,since the rotation does not change the angle  $\vartheta$  between  $L_1$  and  $L_2$ , such that  $\cos \vartheta = L_1 \cdot L_2$ . Under isotropy assumption the joint distribution of  $X(L_1)$  and  $X(L_2)$  equals to the joint distribution of X(N) and  $X(g_{L_2L_1}L_1)$ , i.e. X(N) and  $X(\vartheta, 0)$  contain all pairwise information. In other words the covariance of an isotropic field depends on  $\vartheta$  only,  $\operatorname{Cum}_2(X(L_1), X(L_2)) = \operatorname{Cov}(X(L_1), X(L_2)) = \mathcal{C}_2(g_{L_2L_1}L_1, N)$ , necessarily

 $C_2(L_1, L_2) = C_2(L_1 \cdot L_2) = C(\cos \vartheta)$ . Now

$$X(N) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell}^{m}(N) Z_{\ell}^{m}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \delta_{m,0} \sqrt{\frac{2\ell+1}{4\pi}} Z_{\ell}^{m}$$

$$= \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} Z_{\ell}^{0},$$

and

$$X(g_{L_{2}L_{1}}L_{1}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell}^{m}(g_{L_{2}L_{1}}L_{1}) Z_{\ell}^{m},$$

We have  $\operatorname{Cum}_2(Z_\ell^0, Z_k^n) = \delta_{\ell,k} \delta_{0,n} f_\ell$ , hence

$$C_{2}(L_{1}, L_{2}) = \sum_{\ell=0}^{\infty} f_{\ell} \sqrt{\frac{2\ell+1}{4\pi}} Y_{\ell}^{0}(g_{L_{2}L_{1}}L_{1})$$
$$= \sum_{\ell=0}^{\infty} f_{\ell} \frac{2\ell+1}{4\pi} P_{\ell}(\cos \vartheta).$$

Similarly to the time series setup the covariance  $C_2(L_1, L_2)$  is expanded in terms of an orthonormed system  $\frac{2\ell+1}{4\pi}P_{\ell}(\cos\vartheta)$  with coefficients  $f_{\ell}$ . In particular the variance Var(X(L)) is split up into the superposition of  $f_{\ell}$ 's. Hence  $S_2(\ell) = f_{\ell}$  is called **spectrum** of the field X(L) according to the frequency  $\ell$ .

# 4 Bispectrum

If the field X(L) is non-Gaussian then the first characteristic after the second order moments to be considered is the third order cumulant, referred to bicovariance or 3-point covariance also since it is the third order central moment. The corresponding quantity in frequency domain is the bispectrum. Similarly to the spectrum when the covariance (second order cumulant)

$$\operatorname{Cum}_{2}\left(X\left(L_{1}\right),X\left(L_{2}\right)\right) = \sum_{\ell=0}^{\infty} f_{\ell} \frac{2\ell+1}{4\pi} P_{\ell}\left(\cos\vartheta\right),$$

is split up to the superposition of some orthogonal system of functions and the coefficients are called spectrum. We see that the orthogonal and normed system is the Legendre polynomial system  $\left\{\frac{2\ell+1}{4\pi}P_{\ell}\right\}$  and the spectrum  $S_{2}\left(\ell\right)=f_{\ell}$  corresponds to the frequency  $\ell$ . We put a similar question for the higher order angular spectra as well. Namely, under assumption of isotropy of the  $m^{th}$  order cumulant we consider the series expansion according to an

orthonormed system and the coefficients will be called  $m^{th}$  order spectrum. The Wigner 3j symbols

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

will be intensively used from now on, see Appendix B, 12. It depends on the quantum numbers  $\ell_1, \ell_2, \ell_3$  called degrees and orders  $m_1, m_2, m_3$ .

## 4.1 Isotropy in third order

The rotation applied to he triangular array  $Z_{\ell}^{m}$  shows the necessary and sufficient condition for the third order isotropy.

**Lemma 13** The field X(L) is isotropic in third order iff the bicovariance  $\operatorname{Cum}_3\left(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}\right)$  of the triangular series  $Z_{\ell}^m$  has the form

$$\operatorname{Cum}_{3}\left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, Z_{\ell_{3}}^{m_{3}}\right) = \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & m_{3} \end{pmatrix} B_{3}\left(\ell_{1}, \ell_{2}, \ell_{3}\right), \tag{8}$$

with

$$B_3(\ell_1, \ell_2, \ell_3) = \sum_{k_1, k_2, k_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \operatorname{Cum}_3 \left( Z_{\ell_1}^{k_1}, Z_{\ell_2}^{k_2}, Z_{\ell_3}^{k_3} \right).$$

See Appendix C for the proof. Note here that  $\operatorname{Cum}_3\left(Z_{\ell_1}^{m_1},Z_{\ell_2}^{m_2},Z_{\ell_3}^{m_3}\right)=0$ , if  $m_1+m_2+m_3\neq 0$ , that is not all the possible bicovariances come into the picture, moreover the cumulants are depending on the orders  $m_1,m_2,m_3$  through the Wigner 3j symbols only. In other words the third order probabilistic property inside a fixed quantum number  $\ell$  does not show up. The function  $B_3$  of the frequencies is an average of the cumulants  $\operatorname{Cum}_3\left(Z_{\ell_1}^{k_1},Z_{\ell_2}^{k_2},Z_{\ell_3}^{k_3}\right)$  by 'probability' amplitudes, hence it is called as 'angle average bispectrum', we shall turn back to the notion of the bispectrum later in this section.

Notice first that the quantity

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_3 \left( \ell_1, \ell_2, \ell_3 \right), \tag{9}$$

is symmetric in  $\ell_1, \ell_2, \ell_3$ , see Appendix C, proof C. Moreover  $B_3(\ell_1, \ell_2, \ell_3)$  is symmetric in  $\ell_1, \ell_2, \ell_3$  as well, since by parity transformation (7)  $\ell_1 + \ell_2 + \ell_3$  must be even, the coefficients  $\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ k_1 & k_2 & k_3 \end{pmatrix}$  are symmetric as well.

From now on we fix the order of  $\ell_1, \ell_2, \ell_3$  such that  $\ell_1 \leq \ell_2 \leq \ell_3$ .

Now we turn to the angular momentum field  $u_{\ell}$ . First observe

$$\begin{aligned} \operatorname{Cum}_{3}\left(u_{\ell_{1}}\left(L_{1}\right),u_{\ell_{2}}\left(L_{2}\right),u_{\ell_{3}}\left(L_{3}\right)\right) &= \sum_{m_{1},m_{2},m_{3}} Y_{\ell_{1}}^{m_{1}}\left(L_{1}\right) Y_{\ell_{2}}^{m_{2}}\left(L_{2}\right) Y_{\ell_{3}}^{m_{3}}\left(L_{3}\right) \operatorname{Cum}_{3}\left(Z_{\ell_{1:3}}^{m_{1:3}}\right) \\ &= \sum_{m_{1},m_{2},m_{3}} Y_{\ell_{1}}^{m_{1}}\left(L_{1}\right) Y_{\ell_{2}}^{m_{2}}\left(L_{2}\right) Y_{\ell_{3}}^{m_{3}}\left(L_{3}\right) \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & m_{3} \end{pmatrix} B_{3}\left(\ell_{1},\ell_{2},\ell_{3}\right) \\ &= B_{3}\left(\ell_{1},\ell_{2},\ell_{3}\right) \sum_{m_{1},m_{2},m_{3}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & m_{3} \end{pmatrix} Y_{\ell_{1}}^{m_{1}}\left(L_{1}\right) Y_{\ell_{2}}^{m_{2}}\left(L_{2}\right) Y_{\ell_{3}}^{m_{3}}\left(L_{3}\right). \end{aligned}$$

This last expression is invariant under both the rotation of  $L_j$ 's and ordering of  $\ell_1, \ell_2, \ell_3$ . The 3-product of spherical harmonics  $Y_{\ell}^m$ ,

$$\widetilde{I}_{\ell_{1},\ell_{2},\ell_{3}}\left(L_{1},L_{2},L_{3}\right) = \sum_{m_{1},m_{2},m_{3}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & m_{3} \end{pmatrix} Y_{\ell_{1}}^{m_{1}}\left(L_{1}\right) Y_{\ell_{2}}^{m_{2}}\left(L_{2}\right) Y_{\ell_{3}}^{m_{3}}\left(L_{3}\right)$$

is rotational invariant, see Appendix (32), therefore without any loss of generality we apply the rotation  $g_{L_3L_2}$  which takes the location  $L_3$  into the North pole N and at the same time takes  $L_2$  into the zOx- plane

$$\begin{aligned} & \operatorname{Cum}_{3}\left(u_{\ell_{1}}\left(L_{1}\right), u_{\ell_{2}}\left(L_{2}\right), u_{\ell_{3}}\left(L_{3}\right)\right) = \operatorname{Cum}_{3}\left(u_{\ell_{1}}\left(g_{L_{3}L_{2}}L_{1}\right), u_{\ell_{2}}\left(g_{L_{3}L_{2}}L_{2}\right), u_{\ell_{3}}\left(N\right)\right) \\ & = B_{3}\left(\ell_{1}, \ell_{2}, \ell_{3}\right) \sum_{m_{1} = -\ell_{1}}^{\ell_{1}} \sum_{m_{2} = -\ell_{2}}^{\ell_{2}} Y_{\ell_{1}}^{m_{1}}\left(g_{L_{3}L_{2}}L_{1}\right) Y_{\ell_{2}}^{m_{2}}\left(g_{L_{3}L_{2}}L_{2}\right) Y_{\ell_{3}}^{0}\left(N\right) \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & 0 \end{pmatrix} \\ & = \sqrt{\frac{2\ell_{3} + 1}{4\pi}} B_{3}\left(\ell_{1}, \ell_{2}, \ell_{3}\right) \sum_{m_{1}, m_{2}} Y_{\ell_{1}}^{m_{1}}\left(g_{L_{3}L_{2}}L_{1}\right) Y_{\ell_{2}}^{m_{2}}\left(g_{L_{3}L_{2}}L_{2}\right) \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & 0 \end{pmatrix}. \end{aligned}$$

The third order cumulants of  $u_{\ell}$  contains an orthonormed system of functions

$$\begin{split} I_{\ell_{1},\ell_{2},\ell_{3}}\left(L_{1},L_{2},L_{3}\right) &= I_{\ell_{1},\ell_{2},\ell_{3}}\left(g_{L_{3}L_{2}}L_{1},g_{L_{3}L_{2}}L_{2},N\right) \\ &= \sqrt{\frac{2\ell_{3}+1}{4\pi}}\sum_{m_{1}:2}\begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & 0 \end{pmatrix}Y_{\ell_{1}}^{m_{1}}\left(g_{L_{3}L_{2}}L_{1}\right)Y_{\ell_{2}}^{m_{2}}\left(g_{L_{3}L_{2}}L_{2}\right). \end{split}$$

In this way  $I_{\ell_1,\ell_2,\ell_3}(L_1,L_2,L_3)$  is connected to the *bipolar spherical harmonics*, i.e. to the irreducible tensor products of the spherical harmonics with different arguments, [VMK88].

$$\left\{Y_{\ell_{1}}\left(L_{1}\right)\otimes Y_{\ell_{2}}\left(L_{2}\right)\right\}_{\ell,m} = (-1)^{\ell_{1}-\ell_{2}+m}\sqrt{2\ell+1}\sum_{m_{1},m_{2}}\begin{pmatrix}\ell_{1}&\ell_{2}&\ell\\m_{1}&m_{2}&m\end{pmatrix}Y_{\ell_{1}}^{m_{1}}\left(L_{1}\right)Y_{\ell_{2}}^{m_{2}}\left(L_{2}\right)$$

Rewrite  $I_{\ell_1,\ell_2,\ell_3}$  in terms of Euler angles

$$\begin{split} \mathcal{I}_{\ell_{1},\ell_{2},\ell_{3}}\left(\vartheta_{1},\varphi_{1},\vartheta_{2}\right) &= I_{\ell_{1},\ell_{2},\ell_{3}}\left(g_{L_{3}L_{2}}L_{1},g_{L_{3}L_{2}}L_{2},N\right) \\ &= \sqrt{\frac{2\ell_{3}+1}{4\pi}} \sum_{m_{1},m_{2}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & 0 \end{pmatrix} Y_{\ell_{1}}^{m_{1}}\left(\vartheta_{1},\varphi_{1}\right) Y_{\ell_{2}}^{-m_{1}}\left(\vartheta_{2},0\right) \\ &= \sqrt{\frac{2\ell_{3}+1}{4\pi}} \sum_{m_{1},m_{2}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & 0 \end{pmatrix} \left(-1\right)^{m_{1}} Y_{\ell_{1}}^{m_{1}}\left(\vartheta_{1},\varphi_{1}\right) Y_{\ell_{2}}^{m_{1}*}\left(\vartheta_{2},0\right). \end{split}$$

According to the usual measure  $\Omega(dL_1)\Omega(dL_2) = \sin \vartheta_1 d\vartheta_1 d\varphi_1 \sin \vartheta_2 d\vartheta_2 d\varphi_2$  on  $\vartheta \in [0, \pi]$ ,  $\varphi \in [0, 2\pi]$ , we have

$$\int_{\mathbb{S}_2} Y_{\ell}^{m*} (\vartheta, 0) Y_j^k (\vartheta, 0) \Omega (dL) = \delta_{\ell, j} \delta_{m, k},$$

hence

$$\iint_{\mathbb{S}_{2}} \mathcal{I}_{\ell_{1},\ell_{2},\ell_{3}}^{*} \mathcal{I}_{j_{1},j_{2},j_{3}} \Omega \left( dL_{1} \right) \Omega \left( dL_{2} \right) = \delta_{\ell_{1},j_{1}} \delta_{\ell_{2},j_{2}} \delta_{\ell_{3},j_{3}},$$

since

$$(2\ell_3 + 1) \sum_{m_1, m_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & 0 \end{pmatrix}^2 = 1.$$

The third order cumulants of  $u_{\ell}$  simplifies

$$\operatorname{Cum}_{3}\left(u_{\ell_{1}}\left(L_{1}\right),u_{\ell_{2}}\left(L_{2}\right),u_{\ell_{3}}\left(L_{3}\right)\right)=B_{3}\left(\ell_{1},\ell_{2},\ell_{3}\right)\mathcal{I}_{\ell_{1},\ell_{2},\ell_{3}}\left(\vartheta_{1},\varphi_{1},\vartheta_{2}\right).$$

If we denote it by

$$\operatorname{Cum}_{3}(u_{\ell_{1}}(L_{1}), u_{\ell_{2}}(L_{2}), u_{\ell_{2}}(L_{3})) = C_{u_{\ell_{1}}\ell_{2}\ell_{2}}(\vartheta_{1}, \varphi_{1}, \vartheta_{2}),$$

then we have the inversion formula

$$\iint_{\mathbb{S}_{2}} C_{u,j_{1},j_{2},j_{3}} (\vartheta_{1}, \varphi_{1}, \vartheta_{2}) \mathcal{I}_{\ell_{1},\ell_{2},\ell_{3}} (\vartheta_{1}, \varphi_{1}, \vartheta_{2}) \Omega (dL_{1}) \Omega (dL_{2})$$

$$= \delta_{\ell_{1},j_{1}} \delta_{\ell_{2},j_{2}} \delta_{\ell_{3},j_{3}} B_{3} (\ell_{1}, \ell_{2}, \ell_{3}).$$

### 4.2 Bispectrum of isotropic fields on sphere

The use of the bispectrum for detecting non-Gaussianity and nonlinearity is well known in time series analysis [SRG84], [Hin82] [TM98], the similar question has been put and studied for CMB analysis see [Mar04], [AH12] and references therein. The asymptotics of the bispectrum has a long history started by [RvN65], the asymptotic distribution of angular bispectrum when the degrees are increasing is considered by [Mar06], [Mar08].

Start with the general case  $\operatorname{Cum}_3(X(L_1), X(L_2), X(L_3))$ . The value of the bicovariance does not change under the rotation  $g_{L_3L_2}$ 

$$\operatorname{Cum}_{3}(X(L_{1}), X(L_{2}), X(L_{3})) = \operatorname{Cum}_{3}(X(g_{L_{3}L_{2}}L_{1}), X(g_{L_{3}L_{2}}L_{2}), X(N))$$

$$= \operatorname{Cum}_{3}(X(\vartheta_{1}, \varphi_{1}), X(\vartheta_{2}, 0), X(N)),$$

the spherical coordinates of the locations  $g_{L_3L_2}L_1=(\vartheta_1,\varphi_1), g_{L_3L_2}L_2=(\vartheta_2,0)$  are defined by the locations  $L_1, L_2, L_3$  as follows:  $g_{L_3L_2}L_2 \cdot N = L_2 \cdot L_3 = \cos\vartheta_2, g_{L_3L_2}L_1 \cdot N =$  $L_1 \cdot L_3 = \cos\vartheta_1$ . These angles  $(\vartheta_1,\varphi_1,\vartheta_2)$  define uniquely, up to rotations, the triangle given by locations  $L_1, L_2, L_3$ . In general the bicovariance is written  $C_3(\vartheta_1,\varphi_1,\vartheta_2) =$  $\operatorname{Cum}_3(X(\vartheta_1,\varphi_1),X(\vartheta_2,0),X(N))$ , i.e. it depends on a location  $L(\vartheta_2,0)$  from the main circle  $(\varphi_2=0)$  and a general location  $L_0=L_0(\vartheta_1,\varphi_1)$ .

$$\operatorname{Cum}_{3}\left(X\left(L_{1}\right), X\left(L_{2}\right), X\left(L_{3}\right)\right)$$

$$= \sum_{\ell_{1}, \ell_{2}, \ell_{3} = 0}^{\infty} \sqrt{\frac{2\ell_{3} + 1}{4\pi}} \sum_{m_{1} = -\ell_{1}}^{\ell_{1}} \sum_{m_{2} = -\ell_{2}}^{\ell_{2}} \operatorname{Cum}_{3}\left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, Z_{\ell_{3}}^{0}\right) Y_{\ell_{1}}^{m_{1}}\left(g_{L_{3}L_{2}}L_{1}\right) Y_{\ell_{2}}^{m_{2}}\left(g_{L_{3}L_{2}}L_{2}\right)$$

$$= \sum_{\ell_{1}, \ell_{2}, \ell_{3} = 0}^{\infty} B_{3}\left(\ell_{1}, \ell_{2}, \ell_{3}\right) \mathcal{I}_{\ell_{1}, \ell_{2}, \ell_{3}}\left(\vartheta_{1}, \varphi_{1}, \vartheta_{2}\right). \tag{10}$$

This series expansion of  $\operatorname{Cum}_3(X(L_1), X(L_2), X(L_3))$  implies the following definition

**Definition 14** The bispectrum of the isotropic field X(L) is

$$S_3(\ell_1, \ell_2, \ell_3) = B_3(\ell_1, \ell_2, \ell_3),$$

and the bicoherence of X(L) is

$$\frac{S_3(\ell_1, \ell_2, \ell_3)}{\sqrt{S_2(\ell_1) S_2(\ell_2) S_2(\ell_3)}} = \frac{B_3(\ell_1, \ell_2, \ell_3)}{\sqrt{f_{\ell_1} f_{\ell_2} f_{\ell_3}}}.$$

**Theorem 15** The bicovariances  $\operatorname{Cum}_3(X(L_1), X(L_2), X(L_3))$  of the isotropic field X(L) have the series expansion (10) in terms of the bispectrum  $B_3(\ell_1, \ell_2, \ell_3)$  and orthonormed system  $\mathcal{I}_{\ell_1, \ell_2, \ell_3}$ , hence

$$B_{3}\left(\ell_{1},\ell_{2},\ell_{3}\right)=\iint_{\mathbb{S}_{2}}\operatorname{Cum}_{3}\left(X\left(\vartheta_{1},\varphi_{1}\right),X\left(\vartheta_{2},0\right),X\left(N\right)\right)\mathcal{I}_{\ell_{1},\ell_{2},\ell_{3}}\left(\vartheta_{1},\varphi_{1},\vartheta_{2}\right)\Omega\left(dL_{1}\right)\Omega\left(dL_{2}\right).$$

#### 4.2.1 Particular cases

For any locations  $L_1$  and L we have

$$\operatorname{Cum}_{3}\left(u_{\ell_{1}}\left(L_{1}\right),u_{\ell_{2}}\left(L\right),u_{\ell_{3}}\left(L\right)\right)=\sqrt{\prod_{i}\frac{2\ell_{i}+1}{4\pi}}\begin{pmatrix}\ell_{1}&\ell_{2}&\ell_{3}\\0&0&0\end{pmatrix}B_{3}\left(\ell_{1},\ell_{2},\ell_{3}\right)P_{\ell_{1}}\left(\cos L\cdot L_{1}\right),$$

from this readily follows

$$\operatorname{Cum}_{3}(X(L_{1}), X(L), X(L)) = \sum_{\ell_{1}, \ell_{2}, \ell_{3} = 0}^{\infty} \prod_{i=1}^{3} \sqrt{\frac{2\ell_{i} + 1}{4\pi}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ 0 & 0 & 0 \end{pmatrix} \times B_{3}(\ell_{1}, \ell_{2}, \ell_{3}) P_{\ell_{1}}(\cos L \cdot L_{1}).$$

In particular, according to the Condon and Shortley phase convention, we have for any location L the third order central moment

$$\operatorname{Cum}_{3}\left(u_{\ell_{1}}\left(L\right),u_{\ell_{2}}\left(L\right),u_{\ell_{3}}\left(L\right)\right)=\sqrt{\prod\frac{2\ell_{i}+1}{4\pi}}\begin{pmatrix}\ell_{1} & \ell_{2} & \ell_{3}\\ 0 & 0 & 0\end{pmatrix}B_{3}\left(\ell_{1},\ell_{2},\ell_{3}\right),$$

and the third order cumulant (central moment)  $\operatorname{Cum}_{3}(X(L), X(L), X(L))$  of X(L) is

$$\operatorname{Cum}_{3}(X(L), X(L), X(L)) = \operatorname{Cum}_{3}(X(N), X(N), X(N), X(N))$$

$$= \sum_{\ell_{1}, \ell_{2}, \ell_{3} = 0}^{\infty} \operatorname{Cum}_{3}(Z_{\ell_{1}}^{0}, Z_{\ell_{2}}^{0}, Z_{\ell_{3}}^{0}) \prod \sqrt{\frac{2\ell_{i} + 1}{4\pi}}$$

$$= \sum_{\ell_{1}, \ell_{2}, \ell_{3} = 0}^{\infty} \prod \sqrt{\frac{2\ell_{i} + 1}{4\pi}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ 0 & 0 & 0 \end{pmatrix} B_{3}(\ell_{1}, \ell_{2}, \ell_{3})$$

Note that the summation is taken over those  $\ell_1, \ell_2, \ell_3$  when  $\ell_1 + \ell_2 + \ell_3 = \mathcal{L}$  is even and for this case the following explicit expression is valid

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\mathcal{L}/2} \sqrt{\frac{\prod (\mathcal{L} - 2\ell_j)!}{(\mathcal{L} + 1)!}} \frac{(\mathcal{L}/2)!}{\prod (\mathcal{L}/2 - \ell_j)!}$$

### 4.3 Linear field

First we assume that the rows of the triangle array  $Z_{\ell}^m$  contain independent variables. In other words the angular momentum field  $u_{\ell}$  is linear. Then for a fixed degree  $\ell$ 

$$\operatorname{Cum}_{3}\left(Z_{\ell}^{m_{1}}, Z_{\ell}^{m_{2}}, Z_{\ell}^{m_{3}}\right) = \delta_{m_{i}=m} \begin{pmatrix} \ell & \ell & \ell \\ m & m & m \end{pmatrix} B_{3}\left(\ell, \ell, \ell\right),$$

now the selection rules apply, see Appendix B, 13

$$\operatorname{Cum}_{3}\left(Z_{\ell}^{m_{1}}, Z_{\ell}^{m_{2}}, Z_{\ell}^{m_{3}}\right) = \delta_{m_{i}=m}\delta_{m=0}\begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix} B_{3}\left(\ell, \ell, \ell\right).$$

Hence the only nonzero third order cumulant might be  $\operatorname{Cum}_3(Z_\ell^0, Z_\ell^0, Z_\ell^0)$ . Further the bispectrum

$$B_3(\ell, \ell, \ell) = \sum_{k} \begin{pmatrix} \ell & \ell & \ell \\ k & k & k \end{pmatrix} \operatorname{Cum}_3 \left( Z_{\ell}^k, Z_{\ell}^k, Z_{\ell}^k \right)$$
$$= \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix} \operatorname{Cum}_3 \left( Z_{\ell}^0, Z_{\ell}^0, Z_{\ell}^0 \right),$$

therefore

$$\operatorname{Cum}_{3}\left(Z_{\ell}^{0}, Z_{\ell}^{0}, Z_{\ell}^{0}\right) = \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}^{2} \operatorname{Cum}_{3}\left(Z_{\ell}^{0}, Z_{\ell}^{0}, Z_{\ell}^{0}\right).$$

We conclude from this that from the isotropy and independence follows  $\operatorname{Cum}_3(Z_\ell^m, Z_\ell^m, Z_\ell^m) = 0$ . Moreover if all the members of the triangle array  $Z_\ell^m$  are independent then  $\operatorname{Cum}_3(X(L_1), X(L_2), X(L_3)) = 0$ . Third order cumulants vanish for instance when the distribution is Gaussian, or symmetric, etc.

## 4.4 Symmetries of the bispectrum

The cumulants  $\operatorname{Cum}_3(X(L_1), X(L_2), X(L_3))$  according to different locations  $L_1, L_2$  and  $L_3$  are defined by the spherical triangle with vertices  $L_1, L_2$  and  $L_3$ . The efficiency of computations and avoiding the redundancy in statistical procedures require to determine the principal domains. The *principal domain* for the bispectrum according to the frequencies  $\ell_1, \ell_2, \ell_3$  is the following

- 1.  $\ell_1, \ell_2, \ell_3$  is monotone:  $\ell_1 \leq \ell_2 \leq \ell_3$ ,
- 2.  $\ell_1 + \ell_2 + \ell_3$  is even, see Remark 11,
- 3.  $\ell_1, \ell_2, \ell_3$  fulfils the triangular inequality  $|\ell_1 \ell_2| \le \ell_3 \le \ell_1 + \ell_2$ ,
- 4.  $\operatorname{Cum}_3\left(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}\right) = 0$ , unless  $m_1 + m_2 + m_3 = 0$ .

Similarly the principal domain for the bicovariance according to the locations  $(L_1, L_2, L_3)$  is  $\{(\vartheta_1, \varphi_1, \vartheta_2) | \vartheta_1 \in [0, \pi], \varphi_2 \in [0, \pi], \vartheta_2 \in [0, \pi]\}$ . We apply the following notation of these angles  $\cos \vartheta_1 = L_2 \cdot L_3$ ,  $\cos \vartheta_2 = L_1 \cdot L_3$ , and  $\varphi_2 = \varphi_3$  is the surface angle at  $L_3$ . The third central angle is given by  $\cos \vartheta_3 = L_1 \cdot L_2$ . The surface angle  $\varphi_3$  can be calculated for instance from the cosine formula

$$\cos \theta_3 = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \varphi_2$$
.

## 5 Trispectrum

### 5.1 Isotropy in fourth order

In the papers [Hu01], [KK06] for instance, the notion of trispectrum is used for the coefficients of the series expansion of the fourth order moments. Naturally, because of the expression of the moment via cumulants (21) it contains terms which correspond to the Gaussian part of the field. This is one of the reasons that the time series analysis considers the series expansion of cumulants instead of moments and reserve the notion of trispectrum for this case only, see [Bri65], [MH95]. The cumulants up to third order equal to the central moments, but it is not so for higher order, see Appendix A. Cumulants are very useful, for instance they measure the distance of a distribution from the Gaussian one.

**Lemma 16** The field X(L) is isotropic in fourth order iff the cumulant  $\operatorname{Cum}_4\left(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}, Z_{\ell_4}^{m_4}\right)$  of the triangular array  $Z_{\ell}^m$  has the form

$$\operatorname{Cum}_{4}\left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, Z_{\ell_{3}}^{m_{3}}, Z_{\ell_{4}}^{m_{4}}\right) = \sum_{\ell^{1}, m^{1}} (-1)^{m^{1}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell^{1} \\ m_{1} & m_{2} & m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell_{4} \\ -m^{1} & m_{3} & m_{4} \end{pmatrix} \times \sqrt{2\ell^{1} + 1} T_{4} \left(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} | \ell^{1}\right)$$

$$(11)$$

where  $m^1 = -m_1 - m_2$ .

See the Appendix D for the proof. The function  $T_4\left(\ell_1,\ell_2,\ell_3,\ell_4|\ell^1\right)$  has the form

$$T_{4}\left(\ell_{1},\ell_{2},\ell_{3},\ell_{4}|\ell^{1}\right) = \sqrt{2\ell^{1}+1} \sum_{k_{1},k_{2},k_{3},k_{4},k^{1}} (-1)^{k^{1}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell^{1} \\ k_{1} & k_{2} & k^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell_{4} \\ -k^{1} & k_{3} & k_{4} \end{pmatrix} \times \operatorname{Cum}_{4}\left(Z_{\ell_{1}}^{k_{1}}, Z_{\ell_{2}}^{k_{2}}, Z_{\ell_{3}}^{k_{3}}, Z_{\ell_{4}}^{k_{4}}\right)$$

$$(12)$$

where  $k^1 = -k_1 - k_2$ .

We notice here, the (12) shows that the cumulants of  $Z_{\ell}^{m}$  and the function  $T_{4}\left(\ell_{1},\ell_{2},\ell_{3},\ell_{4}|\ell^{1}\right)$  define each other uniquely.

**Remark 17** If we assume that  $T_4(\ell_1, \ell_2, \ell_3, \ell_4 | \ell^1)$  is known for  $\ell_1 \leq \ell_2 \leq \ell_3 \leq \ell_4$ , then the cumulants  $Cum_4(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}, Z_{\ell_4}^{m_4})$  are given for any  $(\ell_1, \ell_2, \ell_3, \ell_4)$ . Vice versa  $T_4(\ell_1, \ell_2, \ell_3, \ell_4 | \ell^1)$  can be calculated for any  $(\ell_1, \ell_2, \ell_3, \ell_4)$  by (12).

The sum in (12) works for  $k_1, k_2, k_3, k_4$  and  $k^1 = k_1 + k_2 = -k_3 - k_4$ . Hence the summation contains those  $k_m$ 's when equation  $k_1 + k_2 + k_3 + k_4 = 0$ , fulfils. Similarly  $\operatorname{Cum}_4\left(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}, Z_{\ell_4}^{m_4}\right) = 0$ , if  $m_1 + m_2 + m_3 + m_4 \neq 0$ . The triangular inequality for  $\ell_1, \ell_2, \ell^1$  and  $\ell_3, \ell_4, \ell^1$  suggests that the 'quadrilateral' with edges  $(\ell_1, \ell_2, \ell_3, \ell_4)$  consists of two triangles  $\ell_1, \ell_2, \ell^1$  and  $\ell_3, \ell_4, \ell^1$ . The cumulants are invariant of the order of their variables therefore this should be true for the other diagonal as well  $\ell_4, \ell_1, \ell$  and  $\ell_2, \ell_3, \ell$ . Actually since the left hand side of (11) is symmetric in  $\ell_1, \ell_2, \ell_3, \ell_4$ , the right hand side does not depend of the choice of the diagonal.

Note that parity transformation (7) implies that  $\ell_1 + \ell_2 + \ell_3 + \ell_4$  must be even and we turn to the field  $u_\ell$ , see 6,

$$\operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(L_{1}\right), u_{\ell_{2}}\left(L_{2}\right), u_{\ell_{3}}\left(L_{3}\right), u_{\ell_{4}}\left(L_{4}\right)\right)$$

$$= \sum_{m_{1:4}} \prod_{j=1}^{4} Y_{\ell_{j}}^{m_{j}}\left(L_{j}\right) \operatorname{Cum}_{4}\left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, Z_{\ell_{3}}^{m_{3}}, Z_{\ell_{4}}^{m_{4}}\right)$$

$$= \sum_{\ell_{1}, m_{1}} T_{4}\left(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} | \ell^{1}\right) \sqrt{2\ell^{1} + 1} \sum_{m_{1:4}} \prod_{j=1}^{4} Y_{\ell_{j}}^{m_{j}}\left(L_{j}\right) \begin{pmatrix} \ell_{1} & \ell_{2} & \ell^{1} \\ m_{1} & m_{2} & -m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell_{4} \\ m^{1} & m_{3} & m_{4} \end{pmatrix}.$$

The cumulant in the left hand side is symmetric in  $\ell_1, \ell_2, \ell_3, \ell_4$ , and invariant under rotation. The function

$$\begin{split} \widetilde{I}_{\ell_{1},\ell_{2},\ell_{3},\ell_{4},\ell^{1}}\left(L_{1},L_{2},L_{3},L_{4}\right) &= \sqrt{2\ell^{1}+1} \sum_{m_{1},\ldots m_{4},m^{1}} \prod_{j=1}^{4} Y_{\ell_{j}}^{m_{j}}\left(L_{j}\right) \\ &\times \begin{pmatrix} \ell_{1} & \ell_{2} & \ell^{1} \\ m_{1} & m_{2} & -m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell_{4} \\ m^{1} & m_{3} & m_{4} \end{pmatrix}, \end{split}$$

is invariant under the rotation, since if we apply a rotation g on each  $L_j$ , then we can use Lemma 27 Appendix to show that the right hand side does not change. Therefore we apply the rotation  $g_{L_4L_3}$ , which takes the location  $L_4$  to the North pole and at the same time takes  $L_3$  into the plane xOz,

$$\begin{aligned} \operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(L_{1}\right), u_{\ell_{2}}\left(L_{2}\right), u_{\ell_{3}}\left(L_{3}\right), u_{\ell_{4}}\left(L_{4}\right)\right) \\ &= \operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(g_{L_{4}L_{3}}L_{1}\right), u_{\ell_{2}}\left(g_{L_{4}L_{3}}L_{2}\right), u_{\ell_{3}}\left(g_{L_{4}L_{3}}L_{3}\right), u_{\ell_{4}}\left(N\right)\right) \\ &= \sqrt{\frac{2\ell_{4}+1}{4\pi}} \sum_{\ell^{1}} T_{4}\left(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} \middle| \ell^{1}\right) \sqrt{2\ell^{1}+1} \\ &\times \sum_{m_{1}, m_{2}, m_{3}, m^{1}} \prod_{j=1}^{3} Y_{\ell_{j}}^{m_{j}}\left(g_{L_{4}L_{3}}L_{j}\right) \begin{pmatrix} \ell_{1} & \ell_{2} & \ell^{1} \\ m_{1} & m_{2} & -m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell_{4} \\ m^{1} & m_{3} & 0 \end{pmatrix}, \end{aligned}$$

we have

$$\operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(L_{1}\right), u_{\ell_{2}}\left(L_{2}\right), u_{\ell_{3}}\left(L_{3}\right), u_{\ell_{4}}\left(L_{4}\right)\right) = \sum_{\ell_{1}} T_{4}\left(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} | \ell^{1}\right) \mathcal{I}_{\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4}, \ell^{1}}\left(\vartheta_{1}, \vartheta_{2}, \vartheta_{3}, \varphi_{1}, \varphi_{2}\right),$$

where the function  $\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4,\ell^1}$  is given by

$$\mathcal{I}_{\ell_{1},\ell_{2},\ell_{3},\ell_{4},\ell^{1}}(\vartheta_{1},\vartheta_{2},\vartheta_{3},\varphi_{1},\varphi_{2}) = \sqrt{\frac{2\ell_{4}+1}{4\pi}(2\ell^{1}+1)} \sum_{m_{1:3},m^{1}} \prod_{j=1}^{3} Y_{\ell_{j}}^{m_{j}}(g_{L_{4}L_{3}}L_{j}) \times \begin{pmatrix} \ell_{1} & \ell_{2} & \ell^{1} \\ m_{1} & m_{2} & -m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell_{4} \\ m^{1} & m_{3} & 0 \end{pmatrix}.$$

Actually  $\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4,\ell^1}$  is the rotated version of  $\widetilde{I}_{\ell_1,\ell_2,\ell_3,\ell_4,\ell^1}$ , it corresponds to the tripolar spherical harmonics defined as irreducible tensor products of the spherical harmonics with different arguments [VMK88], p.160. We define the spherical coordinates  $(\vartheta_1,\vartheta_2,\vartheta_3,\varphi_1,\varphi_2)$  as follows  $g_{L_4L_3}L_3=(\vartheta_3,0),\ g_{L_4L_3}L_j=(\vartheta_j,\varphi_j),\ j=1,2.$  Note the orthonormality of the system according to the measure  $\prod_{k=1}^3\Omega\left(dL_k\right)=\prod_{k=1}^3\sin\vartheta_k d\vartheta_k d\varphi_k,\ \vartheta_k\in[0,\pi],\ \varphi_k\in[0,2\pi]$ , i.e.

$$\iiint_{\mathbb{S}_2} \mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4,\ell^1}^* \mathcal{I}_{j_1,j_2,j_3,j_4,\ell^2} \prod_{k=1}^3 \Omega\left(dL_k\right) = \delta_{\ell^1,\ell^2} \prod_{k=1:4} \delta_{\ell_k,j_k},$$

since collecting the coefficients after integration, we have

$$(2\ell_4 + 1) \sum_{m^1, m_3} \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ m^1 & m_3 & 0 \end{pmatrix}^2 (2\ell^1 + 1) \sum_{m_1, m_2, m^2} \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ m_1 & m_2 & -m^2 \end{pmatrix}^2 = 1.$$

The expression for the  $\operatorname{Cum}_4(X(L_1), X(L_2), X(L_3), X(L_4)) = \mathcal{C}_4(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2)$  is straightforward

$$C_{4}(\vartheta_{1},\vartheta_{2},\vartheta_{3},\varphi_{1},\varphi_{2}) = \sum_{\ell_{1},\ell_{2},\ell_{3},\ell_{4}=0}^{\infty} \operatorname{Cum}_{4}(u_{\ell_{1}}(L_{1}),\ldots,u_{\ell_{4}}(L_{4}))$$

$$= \sum_{\ell_{1},\ldots,\ell_{4}=0}^{\infty} \sum_{\ell^{1}} T_{4}(\ell_{1},\ell_{2},\ell_{3},\ell_{4}|\ell^{1}) \mathcal{I}_{\ell_{1},\ell_{2},\ell_{3},\ell_{4},\ell^{1}}(L_{1},\ldots,L_{4}).$$
(13)

**Definition 18** The trispectrum of the isotropic field X(L) is given by

$$S_4(\ell_1, \ell_2, \ell_3, \ell_4 | \ell^1) = T_4(\ell_1, \ell_2, \ell_3, \ell_4 | \ell^1).$$

**Theorem 19** The fourth order cumulant  $\operatorname{Cum}_4(X(L_1), X(L_2), X(L_3), X(L_4))$  of the isotropic field X(L) have the series expansion (13) in terms of the trispectrum  $S_4(\ell_1, \ell_2, \ell_3, \ell_4 | \ell^1)$  and orthonormed system  $\mathcal{I}_{\ell_1, \ell_2, \ell_3, \ell_4, \ell^1}$ , hence

$$S_4\left(\ell_1,\ell_2,\ell_3,\ell_4|\,\ell^1\right) = \iiint\limits_{\mathbb{S}_2} \mathcal{C}_4\left(\vartheta_1,\vartheta_2,\vartheta_3,\varphi_1,\varphi_2\right) \mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4,\ell^1}\left(\vartheta_1,\vartheta_2,\vartheta_3,\varphi_1,\varphi_2\right) \prod\limits_{k=1}^3 \Omega\left(dL_k\right).$$

In particular, we have formulae for the marginal fourth order cumulants

$$\operatorname{Cum}_{4}(X(L_{1}), X(L_{2}), X(L), X(L)) = \sum_{\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} = 0}^{\infty} \sqrt{\frac{2\ell_{3} + 1}{4\pi}} \sum_{\ell^{1}} \begin{pmatrix} \ell_{3} & \ell_{4} & \ell^{1} \\ 0 & 0 & 0 \end{pmatrix} \times T_{4}(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} | \ell^{1}) \mathcal{I}_{\ell_{1}, \ell_{2}, \ell^{1}}(\vartheta_{1}, \varphi_{1}, \vartheta_{2}),$$

and

 $Cum_4(X(L_1), X(L), X(L), X(L))$ 

$$= \sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty} \sqrt{\prod_{j=1}^{4} \frac{2\ell_j+1}{4\pi}} \sum_{\ell^1} T_4 \left(\ell_1, \dots, \ell_4 | \ell^1\right) \times \sqrt{2\ell^1+1} \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ 0 & 0 & 0 \end{pmatrix} P_{\ell_1} \left(L \cdot L_1\right)$$

also

 $\operatorname{Cum}_{4}\left(X\left(L\right),X\left(L\right),X\left(L\right),X\left(L\right)\right)$ 

$$= \sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty} \sqrt{\prod_{j=1}^{4} \frac{2\ell_i+1}{4\pi}} \sum_{\ell^1} T_4 \left(\ell_1,\ell_2,\ell_3,\ell_4|\ell^1\right) \sqrt{2\ell^1+1} \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ 0 & 0 & 0 \end{pmatrix}.$$

#### 5.2 Linear field

If the triangular array  $Z_{\ell}^m$  contains independent variables, i.e. the angular momentum field  $u_{\ell}$  is linear. Then for a fixed degree  $\ell$ 

$$\operatorname{Cum}_{4}\left(Z_{\ell}^{m_{1}}, Z_{\ell}^{m_{2}}, Z_{\ell}^{m_{3}}, Z_{\ell}^{m_{4}}\right) = \delta_{m_{i}=m} \sum_{\ell^{1}, m^{1}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell^{1} \\ m_{1} & m_{2} & m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell_{4} \\ -m^{1} & m_{3} & m_{4} \end{pmatrix} \times (-1)^{m^{1}} \sqrt{2\ell^{1} + 1} T_{4} \left(\ell, \ell, \ell, \ell \mid \ell^{1}\right),$$

from the selection rules follows  $m^1 = -(m_1 + m_2) = -2m$ ,  $m^1 = m_3 + m_4 = 2m$ , therefore  $m_i = 0$ . Now for similar reason

$$T_4(\ell,\ell,\ell,\ell|\ell^1) = \sqrt{2\ell^1 + 1} \operatorname{Cum}_4(Z_{\ell}^0, Z_{\ell}^0, Z_{\ell}^0, Z_{\ell}^0) \begin{pmatrix} \ell & \ell & \ell^1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}.$$

The only nonzero cumulants are

$$\operatorname{Cum}_{4}\left(Z_{\ell}^{0}, Z_{\ell}^{0}, Z_{\ell}^{0}, Z_{\ell}^{0}\right) = \sum_{\ell^{1}} \sqrt{2\ell^{1} + 1} \begin{pmatrix} \ell & \ell & \ell^{1} \\ 0 & 0 & 0 \end{pmatrix}^{2} \begin{pmatrix} \ell^{1} & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}^{2} \operatorname{Cum}_{4}\left(Z_{\ell}^{0}, Z_{\ell}^{0}, Z_{\ell}^{0}, Z_{\ell}^{0}\right),$$

hence for an isotropic linear field  $u_{\ell}$  Cum<sub>4</sub>  $\left(Z_{\ell}^{m_1}, Z_{\ell}^{m_2}, Z_{\ell}^{m_3}, Z_{\ell}^{m_4}\right) = 0$ , for all  $\ell$  and  $m_i$ . If additionally the series of  $u_{\ell}$  is independent then Cum<sub>4</sub>  $\left(X\left(L_1\right), X\left(L_2\right), X\left(L_3\right), X\left(L_4\right)\right) = 0$  follows.

# 6 Higher order spectra for isotropic fields

The generalization of the bispectrum and trispectrum is possible for arbitrary higher order [MP10], [MP11]. If the bispectrum and trispectrum are zero then the field can not be Gaussian, in case all polyspectra, except the second order one, are zero then the isotropic field is necessary Gaussian. First we show that the characterization of the isotropy in  $p^{th}$  order can be given similarly to the previous cases.

Lemma 20 The field

$$X\left(L\right) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell}^{m} Y_{\ell}^{m}\left(L\right),$$

is isotropic in  $p^{th}$  order (p > 3) iff the cumulant  $\operatorname{Cum}_p\left(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, \dots, Z_{\ell_p}^{m_p}\right)$  of the triangular array  $Z_{\ell}^m$  has the form

$$\operatorname{Cum}_{p}\left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, \dots, Z_{\ell_{p}}^{m_{p}}\right) = \sum_{\substack{\ell^{1}, \dots, \ell^{p-3} \\ m^{1}, \dots, m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^{a}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^{a} + 1} \widetilde{S}_{p}\left(\ell_{1}, \dots, \ell_{p} | \ell^{1}, \dots, \ell^{p-3}\right),$$

$$(14)$$

where  $\ell^0 = \ell_1$ ,  $\ell^{p-2} = \ell_p$ ,  $k^0 = -k_1$ ,  $k^{p-2} = k_p$ ,  $m^0 = -m_1$ ,  $m^{p-2} = m_p$ . The function  $\widetilde{S}_p\left(\ell_1,\ldots,\ell_p|\ell^1,\ldots,\ell^{p-3}\right)$  has the form

$$\widetilde{S}_{p}\left(\ell_{1},\ldots,\ell_{p}|\ell^{1},\ldots,\ell^{p-3}\right) = \sum_{\substack{k_{1},\ldots,k_{p}\\k^{1},\ldots,k^{p-3}}} (-1)^{\sum_{a=1}^{p-3}k^{a}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1}\\ -k^{a} & k_{a+2} & k^{a+1} \end{pmatrix} \times \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \operatorname{Cum}_{p}\left(Z_{\ell_{1}}^{k_{1}}, Z_{\ell_{2}}^{k_{2}}, \ldots, Z_{\ell_{p}}^{k_{p}}\right). \tag{15}$$

See the Appendix E for the proof. Consider a convex polygon with vertices  $A_{1:p-1}$ , and edges  $\ell_{1:p}$ . The diagonals denoted by  $\ell^j$ ,  $j=1,2,\ldots,p-3$ , starting from the vertex  $A_1$  divide this polygon into p-2 triangles. The first triangle has sides (angular mometum)  $\ell_1$ ,  $\ell_2$  and  $\ell^1$ , the next one has sides  $\ell^1$ ,  $\ell_3$  and  $\ell^2$ , the general one is  $\ell^a$ ,  $\ell_{a+2}$  and  $\ell^{a+1}$ , finally the last one  $\ell^{p-3}$ ,  $\ell_{p-1}$  and  $\ell_p$ . For each a the sides of the triangle  $(\ell^a, \ell_{a+2}, \ell^{a+1})$  should fulfil the triangle inequality  $|\ell^a - \ell^{a+1}| \leq \ell_{a+2} \leq \ell^a + \ell^{a+1}$ . The coefficients in (14) will differ from zero if orders  $-m^a$ ,  $m_{a+2}$ ,  $m^{a+1}$ , fulfil the assumption  $-m^a + m_{a+2} + m^{a+1} = 0$ , for all a. This implies  $m_1 + m_2 = -m^1$ ,  $m_3 + m^2 = m^1$ , ...,  $m_{p-1} + m_p = m^{p-3}$ . Let us plug in consecutively  $m^a$  and we shall arrive to the result  $m_1 + m_2 + \cdots + m_p = 0$ . Hence  $\operatorname{Cum}_p\left(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, \ldots, Z_{\ell_p}^{m_p}\right) = 0$ , unless  $m_1 + m_2 + \cdots + m_p = 0$ .

**Remark 21** The cumulant  $\operatorname{Cum}_p\left(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, \dots, Z_{\ell_p}^{m_p}\right)$  is invariant under the order of the quantum numbers  $\ell_1, \ell_2, \dots, \ell_p$ , hence the right hand side of (14) is invariant as well. The result is that the function  $\widetilde{S}_p\left(\ell_1, \dots, \ell_p | \ell^1, \dots, \ell^{p-3}\right)$  given on values  $\ell_1 \leq \ell_2 \leq \dots \leq \ell_p$  will determine all cumulants  $\operatorname{Cum}_p\left(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, \dots, Z_{\ell_p}^{m_p}\right)$  by (14).

Repeat the representation of the field

$$X\left(L\right) = \sum_{\ell=0}^{\infty} u_{\ell}\left(L\right),\,$$

where

$$u_{\ell}(L) = \sum_{m=-\ell}^{\ell} Y_{\ell}^{m}(L) Z_{\ell}^{m}.$$

We are interested in the invariance of the distribution of  $u_{\ell}(L)$  under rotations. If the location is fixed then  $u_{\ell}(L)$  is connected to the distribution of  $\mathcal{Z}_{\ell}^{k}$ , since the Condon and Shortley phase convention (36) provides

$$u_{\ell}(L) = \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m=-\ell}^{\ell} D_{m,0}^{(\ell)*} Z_{\ell}^{m}.$$

Our main interest is the comparison of the finite dimensional distributions of the process  $u_{\ell}(L)$  to the rotated one in case the rotation carries one location to the North pole N, since  $u_{\ell}$  simplifies to

$$u_{\ell}(N) = \sqrt{\frac{2\ell+1}{4\pi}} Z_{\ell}^{0}.$$

We have

$$\operatorname{Cum}_{p}\left(u_{\ell_{1}}\left(L_{1}\right), u_{\ell_{2}}\left(L_{2}\right), \dots, u_{\ell_{p}}\left(L_{p}\right)\right) = \sum_{m_{1}, \dots, m_{p}} \prod_{j=1}^{p} Y_{\ell_{j}}^{m_{j}}\left(L_{j}\right) \operatorname{Cum}_{p}\left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, \dots, Z_{\ell_{p}}^{m_{p}}\right)$$

$$= \sum_{m_{1}, \dots, m_{p}} \prod_{j=1}^{p} Y_{\ell_{j}}^{m_{j}}\left(L_{j}\right) \sum_{\substack{\ell^{1}, \dots, \ell^{p-3} \\ m^{1}, \dots, m^{p-3}}} \left(-1\right)^{\sum_{a=1}^{p-3} m^{a}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \widetilde{S}_{p}\left(\ell_{1}, \dots, \ell_{p} | \ell^{1}, \dots, \ell^{p-3}\right)$$

$$= \sum_{\ell^{1}, \dots, \ell^{p-3}} \widetilde{I}_{\ell_{1:p}, \ell^{1:p-3}}\left(L_{1}, \dots, L_{p}\right) \widetilde{S}_{p}\left(\ell_{1}, \dots, \ell_{p} | \ell^{1}, \dots, \ell^{p-3}\right).$$

The following p-product of spherical harmonics  $Y_{\ell}^{m}$ 

$$\widetilde{I}_{\ell_1,\dots,\ell_p,\ell^1,\dots,\ell^{p-3}}(L_1,\dots,L_p) = \sum_{\substack{m_1,\dots,m_p\\m^1,\dots,m^{p-3}}} \prod_{j=1}^p Y_{\ell_j}^{m_j}(L_j) \\
\times (-1)^{\sum k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1}\\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1},$$

is rotation invariant, see (32), here  $k^{1:p-3}$  denotes  $(k^1, \ldots, k^{p-3})$  and  $\Sigma k^{1:p-3} = \sum_{j=1}^{p-3} k^j$ , for short. Hence we can apply the rotation  $g_{L_pL_{p-1}}$  such that  $g_{L_pL_{p-1}}L_p = N$ , and  $g_{L_pL_{p-1}}L_{p-1}$  belongs into the plane xOz, we have

$$I_{\ell_{1},\dots,\ell_{p},\ell^{1},\dots,\ell^{p-3}}(L_{1},\dots,L_{p}) = \sqrt{\frac{2\ell_{p}+1}{4\pi}} \sum_{\substack{m_{1},\dots,m_{p}\\m^{1},\dots,m^{p-3}}} \prod_{j=1}^{p-1} Y_{\ell_{j}}^{m_{j}} \left(g_{L_{p}L_{p-1}}L_{j}\right) \times (-1)^{\sum k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1}\\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix} \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \begin{pmatrix} \ell^{p-3} & \ell_{p-1} & \ell_{p}\\ -m^{p-3} & m_{p-1} & 0 \end{pmatrix}.$$

According to the usual measure  $\prod_{k=1}^{p-1} \Omega\left(dL_k\right) = \prod_{k=1}^{p-1} \sin \vartheta_k d\vartheta_k d\varphi_k, \, \vartheta_k \in [0,\pi], \, \varphi_k \in [0,2\pi],$  the system of functions  $I_{\ell_1,\ldots,\ell_p,\ell^1,\ldots,\ell^{p-3}}$  forms an orthogonal system.

Consider now

$$\widetilde{S}_{p}\left(\ell_{1},\ldots,\ell_{p}|\ell^{1},\ldots,\ell^{p-3}\right) = \sum_{\substack{m_{1},\ldots,m_{p}\\m^{1},\ldots,m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^{a}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1}\\m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \operatorname{Cum}_{p}\left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, \ldots, Z_{\ell_{p}}^{m_{p}}\right)$$

where remember  $\ell^0 = \ell_1$ , and  $\ell^{p-2} = \ell_p$ . The left size is symmetric in  $\ell_1, \ldots, \ell_p$  hence the right size must be symmetric as well. One might fix an order for the entries of  $\ell_1, \ldots, \ell_p$  to get a unique representation for the cumulant. We consider a monotone ordering  $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_p$  and refer to it as canonical representation.

Note that parity transformation (7) implies that  $\ell_1 + \ell_2 + \ell_3 + \ldots + \ell_p$  must be even.

$$\operatorname{Cum}_{p}(X(L_{1}), X(L_{2}), X(L_{3}), \dots, X(L_{p})) = \sum_{\ell_{1:p}=0}^{\infty} \operatorname{Cum}_{p}(u_{\ell_{1}}(L_{1}), u_{\ell_{2}}(L_{2}), \dots, u_{\ell_{p}}(L_{p}))$$

$$= \sum_{\ell_{1:p}=0}^{\infty} \widetilde{S}_{p}(\ell_{1}, \dots, \ell_{p} | \ell^{1}, \dots, \ell^{p-3}) I_{\ell_{1}, \dots, \ell_{p}, \ell^{1}, \dots, \ell^{p-3}}(L_{1}, \dots, L_{p}).$$
(16)

**Definition 22** The  $p^{th}$  order polyspectrum of the isotropic field X(L) is  $S_p(\ell_1, \ldots, \ell_p | \ell^1, \ldots, \ell^{p-3}) = \widetilde{S}_p(\ell_1, \ldots, \ell_p | \ell^1, \ldots, \ell^{p-3}).$ 

**Theorem 23** The  $p^{th}$  order cumulant  $\operatorname{Cum}_p(X(L_1), X(L_2), \ldots, X(L_p))$  of the isotropic field X(L) have the series expansion (16) in terms of the polyspectrum  $S_p(\ell_1, \ldots, \ell_p | \ell^1, \ldots, \ell^{p-3})$  and orthonormed system  $\mathcal{I}_{\ell_1, \ldots, \ell_p | \ell^1, \ldots, \ell^{p-3}}$ , hence

$$S_{p}\left(\ell_{1},\ldots,\ell_{p}|\ell^{1},\ldots,\ell^{p-3}\right) = \int \cdots \int_{\mathbb{S}_{2}} C_{p}\left(\vartheta_{1:p-1},\varphi_{1:p-3}\right) \mathcal{I}_{\ell_{1},\ldots,\ell_{p}|\ell^{1},\ldots,\ell^{p-3}}\left(\vartheta_{1:p-1},\varphi_{1:p-3}\right) \prod_{k=1}^{p-1} \Omega\left(dL_{k}\right).$$

## 6.1 Linear field

Let us consider the particular case when  $Z_{\ell}^{m}$  are independent if  $\ell$  is fixed. We have

$$\operatorname{Cum}_{p}\left(Z_{\ell}^{m_{1}}, Z_{\ell}^{m_{2}}, \dots, Z_{\ell}^{m_{p}}\right) = \delta_{m_{i}=m} \sum_{\substack{\ell^{1}, \dots, \ell^{p-3} \\ m^{1}, \dots, m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^{a}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$\times S_{p}\left(\ell \mid \ell^{1}, \dots, \ell^{p-3}\right) \prod_{a=1}^{p-3} \sqrt{2\ell^{a} + 1}.$$

We have seen that for p = 3, 4, all  $m_i = 0$  follows. Let us see the case p = 5,

$$\begin{pmatrix} \ell & \ell^1 \\ m_{1:2} & m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell & \ell^2 \\ -m^1 & m_3 & m^2 \end{pmatrix} \begin{pmatrix} \ell^2 & \ell & \ell \\ -m^2 & m_4 & m_5 \end{pmatrix}$$

then from the selection rules follows that  $m^1 = -(m_1 + m_2) = -2m$ ,  $m^2 = m^1 - m_3 = -3m$ ,  $m^2 = m_4 + m_5 = 2m$ , hence m = 0. In general it is easy to see that  $m^k = -(k+1)m$ , for  $k = 1, 2, \ldots, p-3$ , and at the same time  $m^{p-3} = 2m$ , hence m = 0, and  $m^j = 0$  as well.

Consider the polyspectrum

$$\widetilde{S}_{p}\left(\ell | \ell^{1:p-3}\right) = \sum_{\substack{k_{1}, \dots, k_{p} \\ k^{1} \quad k^{p-3}}} (-1)^{\sum k^{1:p-2}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell & \ell^{a+1} \\ -k^{a} & k_{a+2} & k^{a+1} \end{pmatrix} \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \operatorname{Cum}_{p}\left(Z_{\ell}^{k_{1:p}}\right),$$

from the independence follows that  $\ell_j = \ell$  and  $k_j = k$ , for j = 1, 2, ..., p. Hence similar argument to the previous one leads to the result:  $k_j = 0$ , and  $k^j = 0$  as well. We have

$$\widetilde{S}_p\left(\ell | \ell^{1:p-3}\right) = \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell & \ell^{a+1} \\ 0 & 0 & 0 \end{pmatrix} \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1} \operatorname{Cum}_p\left(Z_\ell^0, Z_\ell^0, \dots, Z_\ell^0\right).$$

Now, by the Lemma 20 we get

$$\operatorname{Cum}_{p}\left(Z_{\ell}^{0}, Z_{\ell}^{0}, \dots, Z_{\ell}^{0}\right) = \sum_{\ell^{1}, \dots, \ell^{p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell & \ell^{a+1} \\ 0 & 0 & 0 \end{pmatrix}^{2} \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \operatorname{Cum}_{p}\left(Z_{\ell}^{0}, Z_{\ell}^{0}, \dots, Z_{\ell}^{0}\right).$$

Hence  $\operatorname{Cum}_p\left(Z_{\ell_1}^{k_1},Z_{\ell_2}^{k_2},\ldots,Z_{\ell_p}^{k_p}\right)=0$ , for all  $k_j$ . Now we have a general conclusion because the only case when all cumulants vanish except the second order one is the Gaussian. Once the rows of  $\{Z_\ell^m\}$  are Gaussian then all the entries of  $\{Z_\ell^m\}$  are independent. Indeed the isotropy implies that all the entries of  $\{Z_\ell^m\}$  are uncorrelated now they are Gaussian hence they are independent.

**Lemma 24** If the isotropic field X(L) is linear, i.e. inside the rows of the generating array  $Z_{\ell}^{m}$  all random variables are independent, then the whole array contains independent entries and X(L) is Gaussian.

# 7 Construction of isotropic field

The stochastic isotropic field

$$X\left(L\right) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell}^{m} Y_{\ell}^{m}\left(L\right),$$

on sphere has very special angular spectra, in particular the form of the cumulants of the triangular array  $\{Z_\ell^m\}$  is restricted, see (14). If one starts with a non-Gaussian continuous in mean square X(L) then the triangular array  $\{Z_\ell^m\}$  is naturally given by the inversion formula

$$Z_{\ell}^{m} = \int_{\mathbb{S}_{2}} X(L) Y_{\ell}^{m*}(L) dL.$$

Now, we address the reverse question of construction of a triangular array  $\{Z_\ell^m\}$  with the desired properties of their cumulants. Let us start with triangular array  $\{\tilde{Z}_\ell^m\}$ , assume it is uncorrelated and all moments exist. Consider the vectors  $\tilde{Z}_\ell = \left[\tilde{Z}_\ell^{-\ell}, \tilde{Z}_\ell^{-\ell+1}, \dots, \tilde{Z}_\ell^{\ell}\right]^{\top}$ ,  $\ell = 0, 1, 2, \dots$ , according to the rows of  $\{\tilde{Z}_\ell^m\}$ . The finite dimensional distribution is characterized by its cumulant function (logarithm of the characteristic function)

$$\Phi_{\widetilde{Z}}\left(\left.\omega_{\ell}\right|\ell=0,1,2,\ldots\right)=\ln\mathsf{E}\exp\left(i\sum_{\ell}\omega_{\ell}^{\top}\widetilde{Z}_{\ell}\right),$$

such that only finite many coordinates of the variables  $(\omega_{\ell}|\ell=0,1,2,\ldots)$  are different from zero. Consider the transformed series  $\widehat{Z}_{\ell}=D^{(\ell)}\widetilde{Z}_{\ell}$ , where  $D^{(\ell)}$  is the Wigner matrix of rotations  $D^{(\ell)}=\left[D_{k,m}^{(\ell)}\right]_{k,m=-\ell}^{\ell}$ , and define a triangular array  $\{Z_{\ell}^m\}$  through the cumulant function

$$\Phi_Z\left(\left.\omega_\ell\right|\ell=0,1,2,\ldots\right) = \int_{SO(3)} \ln \mathsf{E} \exp\left(i\sum_\ell \omega_\ell^\top D^{(\ell)}\widetilde{Z}_\ell\right) dg$$

where again only finite many coordinates of the variables ( $\omega_{\ell}|\ell=0,1,2,\ldots$ ) are different from zero and  $dg=\sin\vartheta d\vartheta d\varphi d\gamma/8\pi^2$ , is the Haar measure with unite mass. This new triangular array  $\{Z_{\ell}^m\}$  will be called Wigner D-transform of  $\{\widetilde{Z}_{\ell}^m\}$ . The third order cumulants of  $Z_{\ell}^m$  for instance

$$\operatorname{Cum}_{3}\left(Z_{\ell_{1}}^{k_{1}}, Z_{\ell_{2}}^{k_{2}}, Z_{\ell_{3}}^{k_{3}}\right) = \int_{SO(3)} \sum_{m_{1}, m_{2}, m_{3}} D_{k_{1}, m_{1}}^{(\ell_{1})} D_{k_{2}, m_{2}}^{(\ell_{2})} D_{k_{3}, m_{3}}^{(\ell_{3})} dg \operatorname{Cum}_{3}\left(\widetilde{Z}_{\ell_{1}}^{m_{1}}, \widetilde{Z}_{\ell_{2}}^{m_{2}}, \widetilde{Z}_{\ell_{3}}^{m_{3}}\right)$$

$$= \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ k_{1} & k_{2} & k_{3} \end{pmatrix} \sum_{m_{1}, m_{2}, m_{3}} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & m_{3} \end{pmatrix} \operatorname{Cum}_{3}\left(\widetilde{Z}_{\ell_{1}}^{m_{1}}, \widetilde{Z}_{\ell_{2}}^{m_{2}}, \widetilde{Z}_{\ell_{3}}^{m_{3}}\right)$$

$$= \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ k_{1} & k_{2} & k_{3} \end{pmatrix} B_{3}\left(\ell_{1}, \ell_{2}, \ell_{3}\right),$$

which fulfils (8). The function  $B_3(\ell_1, \ell_2, \ell_3)$  is given by

$$B_3(\ell_1, \ell_2, \ell_3) = \sum_{m_1, m_2, m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \operatorname{Cum}_3 \left( \widetilde{Z}_{\ell_1}^{m_1}, \widetilde{Z}_{\ell_2}^{m_2}, \widetilde{Z}_{\ell_3}^{m_3} \right),$$

and also

$$B_3(\ell_1, \ell_2, \ell_3) = \sum_{m_1, m_2, m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \operatorname{Cum}_3 \left( Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3} \right).$$

The conclusion is that a subset of cumulants  $\operatorname{Cum}_3\left(\widetilde{Z}_{\ell_1}^{m_1},\widetilde{Z}_{\ell_2}^{m_2},\widetilde{Z}_{\ell_3}^{m_3}\right)$ , i.e.  $m_1+m_2+m_3=0$ , is used in the construction and the superposition with 'probability' amplitudes is applied. In general we also have

$$\operatorname{Cum}_{p}\left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, \dots, Z_{\ell_{p}}^{m_{p}}\right) = \sum_{\substack{\ell^{1}, \dots, \ell^{p-3} \\ m^{1}, \dots, m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^{a}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^{a} + 1} S_{p}\left(\ell_{1:p} | \ell^{1:p-3}\right)$$

where

$$S_{p}\left(\ell_{1:p}|\ell^{1:p-3}\right) = \sum_{\substack{m_{1},\dots,m_{p}\\m^{1},\dots,m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^{a}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1}\\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$
$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \operatorname{Cum}_{p}\left(\widetilde{Z}_{\ell_{1}}^{k_{1}}, \widetilde{Z}_{\ell_{2}}^{k_{2}}, \dots, \widetilde{Z}_{\ell_{p}}^{k_{p}}\right).$$

## A Cumulants

### A.1 Basic properties

Cumulants (also called invariants) are very important quantities for the characterization of a random series, see [RST06] for some details. Moments and cumulants are equivalent, since the moments can be defined as the coefficients in the series expansion of the characteristic function similarly cumulants are the coefficients in the series expansion of the log-characteristic function. The main importance of the cumulants is that it does not contain the Gaussian part of the moments. Let  $\underline{X} \in \mathbb{R}^n$  denote a random vector.

### A.1.1 Symmetry

$$\operatorname{Cum}(\underline{X}) = \operatorname{Cum}(\underline{X}_{\mathfrak{p}(1:n)}), \ \underline{X} \in \mathbb{R}^n,$$

where  $\mathfrak{p}(1:n) = (\mathfrak{p}(1),\mathfrak{p}(2),\ldots,\mathfrak{p}(n)), p \in \mathfrak{P}_{\mathfrak{n}}$  and  $\mathfrak{P}_{\mathfrak{n}}$  denotes the set of all permutations of the integers (1:n).

#### A.1.2 Multilinearity

For any constants  $c_{1:2} = (c_1, c_2)$ 

$$\operatorname{Cum}(c_1Y_1 + c_2Y_2, \underline{X}) = c_1 \operatorname{Cum}(Y_1, \underline{X}) + c_2 \operatorname{Cum}(Y_2, \underline{X}).$$

### A.1.3 Independence

If  $\underline{X} \in \mathbb{R}^n$  is independent of  $\underline{Y} \in \mathbb{R}^m$  where n, m > 0 then

$$\operatorname{Cum}(\underline{X},\underline{Y}) = 0.$$

In particular if n = m then for independent  $\underline{X}$  and  $\underline{Y}$ 

$$\operatorname{Cum}(\underline{X} + \underline{Y}) = \operatorname{Cum}(\underline{X}) + \operatorname{Cum}(\underline{Y}).$$

This formula is the additive version of the formula of the moment of the product of independent variables.

### A.1.4 Gaussianity

The random vector  $\underline{X} \in \mathbb{R}^n$  is Gaussian if and only if for all vector  $k_{(1:m)}$  with elements from (1:n)

$$\operatorname{Cum}(\underline{X}_{k_{(1:m)}}) = 0, \ m > 2.$$

## A.1.5 Expression of the cumulant via moments

The first order cumulants is the moment

$$\operatorname{Cum}(X) = \mathsf{E}X,$$

the second and third order cumulants are the second and third order central moments

$$Cum(X_1, X_2) = Cov(X_1, X_2)$$

$$\tag{17}$$

$$Cum(X_1, X_2, X_3) = \mathsf{E} \prod_{i=1}^{3} (X_i - \mathsf{E} X_i). \tag{18}$$

The general formula is

$$\operatorname{Cum}(\underline{X}) = \sum_{m=1}^{n} (-1)^{m-1} (m-1)! \sum_{U \in \mathcal{P}_n} \prod_{j=1}^{m} \mathsf{E} \ X_{1:n}^{u_{j,1:n}}, \ \underline{X} \in \mathbb{R}^n, \tag{19}$$

where the second summation is taken over all possible partitions U, where  $[u_{j,1:n}]_{j=1:m}$  is an indicator of a partition  $\mathcal{L}$  with m subsets of the set  $(1:n) = (1,2,\ldots,n)$ ,  $u_{j,k} = 1$  if  $k \in \mathcal{L}$ , otherwise 0. The double sum in (19) is over all partitions of  $\mathcal{L} \in \mathcal{P}_n$ , where  $\mathcal{P}_n$  is the set of all partitions of the numbers (1:n).

**Example 25** First note that if n = 3, then the  $m^{th}$  order partitions  $\mathcal{L} \in \mathcal{P}_3$  with the corresponding indicators and permutations of (1:3) are

1. for 
$$m = 1$$
;  $\mathcal{L} = \{(1:3)\}$ 

$$\mathcal{L} \leftrightarrow u_{1,1:3} = \left[1,1,1\right], \quad \mathsf{E} \ X_{1:3}^{u_{1,1:3}} = \mathsf{E} X_1 X_2 X_3,$$

$$\begin{aligned} \mathcal{Z}. \ for \ m = 2; \ \mathcal{L}_1 &= \left\{ (1), \ (2,3) \right\}, \ \mathcal{L}_2 &= \left\{ (1,3), \ (2) \right\}, \ \mathcal{L}_3 &= \left\{ (1,2), \ (3) \right\}, \\ \mathcal{L}_1 &\leftrightarrow \begin{bmatrix} u_{1,1:3} \\ u_{2,1:3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \prod_{j=1}^2 \mathsf{E} \ X_{1:3}^{u_{j,1:3}} &= \mathsf{E} X_1 \mathsf{E} X_2 X_3, \\ \mathcal{L}_2 &\leftrightarrow \begin{bmatrix} u_{1,1:3} \\ u_{2,1:3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \prod_{j=1}^2 \mathsf{E} \ X_{1:3}^{u_{j,1:3}} &= \mathsf{E} X_2 \mathsf{E} X_1 X_3, \\ \mathcal{L}_3 &\leftrightarrow \begin{bmatrix} u_{1,1:3} \\ u_{2,1:3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \prod_{j=1}^2 \mathsf{E} \ X_{1:3}^{u_{j,1:3}} &= \mathsf{E} X_3 \mathsf{E} X_1 X_2, \end{aligned}$$

3. for m = 3;  $\mathcal{L} = \{(1), (2), (3)\}$ 

$$\mathcal{L} \leftrightarrow \begin{bmatrix} u_{1,1:3} \\ u_{2,1:3} \\ u_{2,1:3} \end{bmatrix} = \begin{bmatrix} \ 1 & 0 & 0 \\ \ 0 & 1 & 0 \\ \ 0 & 0 & 1 \end{bmatrix}, \quad \prod_{j=1}^3 \mathsf{E} \ X_{1:3}^{u_{j,1:3}} = \mathsf{E} X_1 \mathsf{E} X_2 \mathsf{E} X_3,$$

Now we apply (19):

$$\begin{aligned} \operatorname{Cum}(X_1, X_2, X_3) &= \mathsf{E} X_1 X_2 X_3 - \mathsf{E} X_1 \mathsf{E} X_2 X_3 - \mathsf{E} X_2 \mathsf{E} X_1 X_3 \\ &- \mathsf{E} X_3 \mathsf{E} X_1 X_2 + 2 \mathsf{E} X_1 \mathsf{E} X_2 \mathsf{E} X_3 \\ &= \mathsf{E} \prod_{i=1}^3 \left( X_i - \mathsf{E} X_i \right), \end{aligned}$$

The first three cumulants, see also (17), equal the central moments but this is not true for higher order cumulants. One might easily check this for the case of cumulants of order four. Indeed,  $\underline{X} \in \mathbb{R}^4$ 

$$\operatorname{Cum}\left(\underline{X}\right) = \operatorname{Cum}\left(\underline{X} - \mathsf{E}\underline{X}\right)$$

$$= \mathsf{E}\prod_{i=1}^{4} \left(X_{i} - \mathsf{E}X_{i}\right) - \operatorname{Cov}\left(X_{1}, X_{2}\right) \operatorname{Cov}\left(X_{3}, X_{4}\right)$$

$$- \operatorname{Cov}\left(X_{1}, X_{3}\right) \operatorname{Cov}\left(X_{2}, X_{4}\right) - \operatorname{Cov}\left(X_{1}, X_{4}\right) \operatorname{Cov}\left(X_{2}, X_{3}\right)$$

$$\neq \mathsf{E}\prod_{i=1}^{4} \left(X_{i} - \mathsf{E}X_{i}\right), \tag{20}$$

unless all covariances are zero.

#### A.1.6 Expression of the moment via cumulants

We consider the moment  $\mathsf{E}\prod_{i=1}^n X_i$  as the general case because the moment  $\mathsf{E}Y^{k_{1:m}}_{1:m}$  can be put into the form  $\mathsf{E}\prod_{i=1}^n X_i$ , where we define  $n=\Sigma k_{(1:m)}$  and

$$\underline{X} = (\underbrace{Y_1, \dots, Y_1}_{k_1}, \dots, \underbrace{Y_m, \dots, Y_m}_{k_m}).$$

Now

$$\mathsf{E}\prod_{i=1}^{n} X_{i} = \sum_{\mathcal{L}\in\mathcal{P}_{n}} \prod_{\mathbf{b}_{i}\in\mathcal{L}} \mathrm{Cum}(X_{\mathbf{b}_{j}}),\tag{21}$$

where the summation is over all partitions  $\mathcal{L} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k\}$  of (1:n). An example of the use of the formula (21) can be seen when n = 3.

**Example 26** Each partition of (1:3) are listed in the previus example therefore if  $\underline{X} \in \mathbb{R}^3$ ,

$$\begin{aligned} \mathsf{E} X_1 X_2 X_3 &= \mathrm{Cum}(X_{1:3}) + \mathrm{Cum}(X_1) \, \mathrm{Cum}(X_2, X_3) \\ &+ \mathrm{Cum}(X_2) \, \mathrm{Cum}(X_1, X_3) + \mathrm{Cum}(X_3) \, \mathrm{Cum}(X_1, X_2) \\ &+ \mathrm{Cum}(X_1) \, \mathrm{Cum}(X_2) \, \mathrm{Cum}(X_3). \end{aligned}$$

Now in particular

$$\mathsf{E} \ X^3 = \mathrm{Cum}(X, X, X) + 3\,\mathrm{Cum}(X)\,\mathrm{Cum}(X, X) + \mathrm{Cum}(X)^3.$$

# **B** Spherical Harmonics

The following notation and results are used in this mauscript.

- 1.  $\mathbb{R}^3$  is Euclidean space of dimension 3.
- 2.  $\mathbb{S}_2$  denotes the **sphere** with radius 1 in  $\mathbb{R}^3$ . The following notations are used; colatitude (angular) coordinate:  $\vartheta \in [0, \pi]$ , longitude coordinate  $\varphi \in [0, 2\pi]$ , North pole: N with  $\vartheta = 0$ , and  $\varphi$  is indeterminate, (latitude coordinate is expressed by colatitude coordinate:  $\pi/2 \vartheta \in [-\pi/2, \pi/2]$ , in that case the North pole has  $\vartheta = \pi/2$ ). The spherical coordinates on  $\mathbb{S}_2$ : ( $\sin \vartheta \cos \varphi$ ,  $\sin \vartheta \sin \varphi$ ,  $\cos \vartheta$ )
- 3. SO(3) denotes the 3D (special orthogonal) rotation group
- 4. Addition Theorem. [EMOT81] vol. 2,7.15 (30), pp.116 Let  $U_1$  and  $U_2$  be two vectors with angle  $\vartheta$  and lengths  $r_1$  and  $r_2$  respectively, and the Euclidean distance is denoted by  $\rho = ||U_1 U_2|| = \sqrt{r_1^2 + r_2^2 2r_1r_2\cos\gamma}$ , then we have

$$\rho^{-\nu} J_{\nu}(\rho) = \left(\frac{r_1 r_2}{2}\right)^{-\nu} \Gamma(\nu) \sum_{\ell=0}^{\infty} (\ell + \nu) C_{\ell}^{\nu}(\cos \gamma) J_{\ell+\nu}(r_1) J_{\ell+\nu}(r_2), \qquad (22)$$

where  $J_{\nu}$  denotes the Bessel function of the first kind,  $C_{\ell}^{\nu}$  Gegenbauer polynomial  $\left(C_{\ell}^{1/2} = P_{\ell}\right)$ .

5. Lapalce-Beltrami operator  $\triangle_B$ ,

$$\Delta_B = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2},$$

$$\Delta_B = \frac{1}{\sin \vartheta} \left[ \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin \vartheta} \frac{\partial^2}{\partial \varphi^2} \right],$$

$$\Delta_B = \frac{\partial^2}{\partial \vartheta^2} + \frac{\cos \vartheta}{\sin \vartheta} \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2},$$

([Apo07], vol. 2 pp. 293).

- 6.  $\nabla$  is the central difference operator  $(\nabla_1^2 + \nabla_2^2) X_{j,k} = X_{j+1,k} + X_{j-1,k} + X_{j,k+1} + X_{j,k-1} 4X_{j,k}$
- 7. Spherical function f of order  $\ell$ .

$$\Delta_B f = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial f}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2 f}{\partial \varphi^2}$$
$$= -\ell (\ell + 1) f,$$

[VMK88]. Regular spherical harmonics  $Y_{\ell}$ , homogeneous polynomial of degree  $\ell$  and  $\nabla^2 Y_{\ell} = 0$ .

8. Normalized Legendre polynomial  $\widetilde{P}_{\ell}$  (Rodrigues' formula)

$$\widetilde{P}_{\ell}(x) = \sqrt{\frac{2\ell+1}{2}} \frac{1}{2^{\ell}\ell!} \frac{d^{\ell}(x^{2}-1)^{\ell}}{dx^{\ell}}, \ x \in [-1,1],$$

$$\int_{-1}^{1} \left(\widetilde{P}_{\ell}(x)\right)^{2} dx = 1$$

Standardized Legendre polynomial  $P_0(x) = 1$ ,

$$P_{\ell}(x) = \frac{1}{2^{\ell}\ell!} \frac{d^{\ell}(x^2 - 1)^{\ell}}{dx^{\ell}}, \ x \in [-1, 1],$$

 $P_{\ell}(1) = 1([\text{EMOT81}], \text{ vol. } 2, \text{ pp.180}) \text{ it is orthogonal and}$ 

$$\int_{-1}^{1} [P_{\ell}(x)]^{2} dx = \frac{2}{2\ell + 1}$$

$$\int_{\mathbb{S}_{2}} [P_{\ell}(\cos \vartheta)]^{2} \Omega (dL) = \int_{0}^{2\pi} \int_{0}^{\pi} [P_{\ell}(\cos \vartheta)]^{2} \sin \vartheta d\vartheta d\varphi$$

$$= 2\pi \int_{0}^{\pi} [P_{\ell}(\cos \vartheta)]^{2} \sin \vartheta d\vartheta$$

$$= 2\pi \int_{-1}^{1} [P_{\ell}(x)]^{2} dx$$

$$= \frac{4\pi}{2\ell + 1}$$
(23)

recurrence

$$(\ell + 1) P_{\ell+1}(x) = (2\ell + 1) x P_{\ell}(x) - \ell P_{\ell-1}(x)$$

generating functions ([EMOT81], vol. 2, pp.183)

$$\sum_{\ell=0}^{\infty} P_{\ell}(x) z^{\ell} = \left(1 - 2xz + z^{2}\right)^{-1/2}, \ x \in (-1, 1), \ |z| < 1$$
 (24)

$$\sum_{\ell=0}^{\infty} \frac{1}{\ell!} P_{\ell} (\cos \gamma) z^{\ell} = e^{z \cos \gamma} J_0 (z \sin \vartheta)$$
(25)

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell + 1/2} P_{\ell}(\cos \gamma) z^{2\ell+1} = F\left(\sin \theta/2, \varphi\right), \quad z = \tan \frac{\varphi}{2}, \quad 0 < \varphi < \frac{\pi}{2}, 0 < \theta < \pi.$$

9.  $P_{\ell}^{m}$  is the associated normalized Legendre function of the first kind (Gegenbauer polynomial at particular indices) of degree  $\ell$  and order m

$$P_{\ell}^{m}(x) = (-1)^{m} \left(1 - x^{2}\right)^{m/2} \frac{d^{m} P_{\ell}(x)}{dx^{m}},$$
$$P_{\ell}^{m}(z) = \left(z^{2} - 1\right)^{m/2} \frac{d^{m} P_{\ell}(z)}{dz^{m}},$$

recurrence formula is

$$(\ell - m + 1) P_{\ell+1}^{m}(x) = (2\ell + 1) x P_{\ell}^{m}(x) - (\ell + m) P_{\ell-1}^{m}(x),$$

in particular  $P_{\ell}^{m}(1) = \delta_{m,0}, P_{\ell}^{0}(x) = P_{\ell}(x),$ 

$$P_{\ell}^{-m}(x) = (-1)^m \frac{\Gamma(\ell - m + 1)}{\Gamma(\ell + m + 1)} P_{\ell}^m(x).$$

10. Funk-Hecke formula ([Ml66]) Suppose G is continuous on [-1,1], then for any spherical harmonic  $Y_{\ell}(L)$ 

$$\int_{\mathbb{S}_{2}} G(L_{1} \cdot L) Y_{\ell}(L) \Omega(dL) = cY_{\ell}(L_{1}),$$

$$c = 2\pi \int_{-1}^{1} G(x) P_{\ell}(x) dx,$$

where  $\Omega(dL) = \sin \vartheta d\vartheta d\varphi$  is Lebesque element of surface area on  $\mathbb{S}_2$ ,  $L_1 \cdot L_2 = \cos \vartheta$ . In particular

$$\int_{\mathbb{S}_{2}} G(L_{1} \cdot L) Y_{\ell}^{m}(L) \Omega(dL) = g_{\ell} Y_{\ell}^{m}(L_{1}),$$

$$g_{\ell} = 2\pi \int_{-1}^{1} G(x) P_{\ell}(x) dx$$
(26)

see [Yad83] pp.72. For a general dimension d instead of  $P_\ell$  one has the Gegenbauer polynomial  $C_\ell^{(d-2)/2}$ . Also

$$\int_{\mathbb{S}_2} G(L_1 \cdot L) P_{\ell}(L_2 \cdot L) \Omega(dL) = f_{\ell} P_{\ell}(L_1 \cdot L_2)$$
$$f_{\ell} = 2\pi \int_{-1}^1 G(x) P_{\ell}(x) dx$$

here  $f_{\ell} = (LT)_{\ell} G$  is the Legendre transform of G. Hence

$$G(\cos \vartheta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \int_{-1}^{1} G(x) P_{\ell}(x) dx P_{\ell}(\cos \vartheta)$$

In particular Funk-Hecke formula when N denotes the north pole

$$\int_{\mathbb{S}_2} \left[ P_{\ell} \left( N \cdot L \right) \right]^2 \Omega \left( dL \right) = c P_{\ell} \left( N \cdot N \right)$$

$$c = 2\pi \int_{-1}^1 \left[ P_{\ell} \left( x \right) \right]^2 dx, \quad P_{\ell} \left( 1 \right) = 1$$

$$\int_{\mathbb{S}_2} \left[ P_{\ell} \left( N \cdot L \right) \right]^2 \Omega \left( dL \right) = \frac{4\pi}{2\ell + 1}$$

11. Orthonormal spherical harmonics with complex values  $Y_{\ell}^{m}(\vartheta,\varphi), \ell=0,1,2,...,$   $m=-\ell,-\ell+1,...-1,0,1,...,\ell-1,\ell$  of degree  $\ell$  and order m (rank  $\ell$  and projection m)

$$Y_{\ell}^{m}(\vartheta,\varphi) = (-1)^{m} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell}^{m}(\cos\vartheta) e^{im\varphi}, \ \varphi \in [0,2\pi], \ \vartheta \in [0,\pi]$$
 (27)

$$Y_{\ell}^{m}(\vartheta,\varphi)^{*} = (-1)^{m} Y_{\ell}^{-m}(\vartheta,\varphi)$$
$$Y_{\ell}^{m}(-\vartheta,-\varphi) = (-1)^{\ell} Y_{\ell}^{m}(\vartheta,\varphi).$$

In particular  $Y_{\ell}^{0}(\vartheta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}(\cos\vartheta), Y_{0}^{0}(\vartheta,\varphi) = \sqrt{\frac{1}{4\pi}},$ 

$$Y_{\ell}^{m}\left(N\right) = \delta_{m,0} \sqrt{\frac{2\ell+1}{4\pi}},$$

we have

$$\int_{0}^{2\pi} \int_{0}^{\pi} |Y_{\ell}^{m}(\vartheta, \varphi)|^{2} \sin \vartheta d\vartheta d\varphi = 1$$

note that  $Y_{\ell}^{m}$  is normalized fully, some authors do not apply  $1/\sqrt{4\pi}$  in the definition of  $Y_{\ell}^{m}$ , also for a sphere with radius R spherical harmonics are normalized additionally  $Y_{\ell}^{m}(\vartheta,\varphi)/R$ . It also follows

$$\begin{split} Y_{\ell}^{m*}\left(\vartheta,\varphi\right) &= Y_{\ell}^{m}\left(\vartheta,-\varphi\right) \\ &= (-1)^{m} Y_{\ell}^{-m}\left(\vartheta,\varphi\right), \\ Y_{\ell}^{-m}\left(\vartheta,\varphi\right) &= (-1)^{m} e^{-i2m\varphi} Y_{\ell}^{m}\left(\vartheta,\varphi\right). \end{split}$$

## 12. Wigner 3*j*-symbols, [Lou06] notation

$$\begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix},$$

Clebsch-Gordan coefficients,

$$C_{\ell_1,k_1;\ell_2,k_2}^{\ell,k} = (-1)^{\ell_1 - \ell_2 + k} \sqrt{2\ell + 1} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix},$$

$$\begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = \frac{(-1)^{\ell_3 + m_3 + 2\ell_1}}{\sqrt{2\ell_3 + 1}} C_{\ell_1,-k_1;\ell_2,-k_2}^{\ell_3,k_3}.$$

Symmetries

$$\begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = (-1)^{\ell_1 + \ell_2 + \ell_3} \begin{pmatrix} \ell_2 & \ell_1 & \ell_3 \\ m_2 & m_1 & m_3 \end{pmatrix} 
= \begin{pmatrix} \ell_2 & \ell_3 & \ell_1 \\ m_2 & m_3 & m_1 \end{pmatrix}, 
\begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = (-1)^{\ell_1 + \ell_2 + \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}.$$
(28)

Orthogonality relations

$$(2\ell+1)\sum_{m_{1:2}} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & m \end{pmatrix} \begin{pmatrix} \ell_{1:2} & j \\ m_{1:2} & k \end{pmatrix} = \delta_{m,k}\delta_{\ell,j}$$
 (29)

$$\sum_{\ell,k} (2\ell+1) \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & k \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & k \end{pmatrix} = \delta_{m_1,k_1} \delta_{m_2,k_2}$$
 (30)

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} = 0,$$

if  $\mathcal{L} = \ell_1 + \ell_2 + \ell_3$  is odd, otherwise see [Edm57], (3.7.17)

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\mathcal{L}/2} \sqrt{\frac{\prod (\mathcal{L} - 2\ell_j)!}{(\mathcal{L} + 1)!}} \frac{(\mathcal{L}/2)!}{\prod (\mathcal{L}/2 - \ell_j)!},$$

if  $\ell$  is even say  $\ell = 2k$ , then

$$\begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{3k} \sqrt{\frac{(\ell!)^3}{(6k+1)!}} \frac{(3k)!}{(k!)^3}.$$

- 13. Selection rules: a Wigner 3j symbol vanishes unless
  - $m_1 + m_2 + m_3 = 0$ ,
  - Integer perimeter rule:  $\mathcal{L} = \ell_1 + \ell_2 + \ell_3$  is an integer (if  $m_1 = m_2 = m_3 = 0$ , then  $\mathcal{L}$  is even).
  - Triangular inequality  $|\ell_1 \ell_2| \le \ell_3 \le \ell_1 + \ell_2$  fulfilles.
- 14. **Gaunt series** [Edm57], (4.6.5)

$$\begin{split} Y_{\ell_{1}}^{m_{1}}\left(\vartheta,\varphi\right)Y_{\ell_{2}}^{m_{2}}\left(\vartheta,\varphi\right) &= \sum_{\ell,m} \sqrt{\frac{\left(2\ell_{1}+1\right)\left(2\ell_{2}+1\right)\left(2\ell+1\right)}{4\pi}} \\ &\times \begin{pmatrix} \ell_{1} & \ell_{2} & \ell \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & m \end{pmatrix} Y_{\ell}^{m*}\left(\vartheta,\varphi\right), \\ Y_{\ell_{1}}^{m_{1}}\left(\vartheta,\varphi\right)Y_{\ell_{2}}^{m_{2}}\left(\vartheta,\varphi\right) &= \sum_{\ell,m} \sqrt{\frac{\left(2\ell_{1}+1\right)\left(2\ell_{2}+1\right)}{4\pi\left(2\ell+1\right)}} C_{\ell_{1},m_{1};\ell_{2},m_{2}}^{\ell,m} C_{\ell_{1},0;\ell_{2},0}^{\ell_{3},0} Y_{\ell}^{m}\left(\vartheta,\varphi\right) \end{split}$$

[VMK88], pp. 144, products of three and more

15. Rayleigh plane wave expansion in 3D:  $\hat{\underline{k}} = \underline{k}/|\underline{k}|$ ,  $\hat{\underline{x}} = \underline{x}/|\underline{x}|$ ,  $r = |\underline{x}|$ ,  $k = |\underline{k}|$ ,

$$e^{i\underline{k}\cdot\underline{x}} = e^{ikr\cos\vartheta} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^{\ell} j_{\ell}\left(kr\right) Y_{\ell}^{m}\left(\widehat{\underline{k}}\right)^{*} Y_{\ell}^{m}\left(\widehat{\underline{x}}\right)$$

where

$$j_{\ell}\left(kr\right) = \sqrt{\frac{\pi}{2kr}} J_{\ell+1/2}\left(kr\right),$$

is the Spherical Bessel function of the first kind, also

$$e^{iz\cos\vartheta} = \sum_{\ell=-\infty}^{\infty} i^{\ell} J_{\ell}(z) e^{i\ell\vartheta},$$

$$e^{iz\cos\vartheta} = J_{0}(z) + \sum_{\ell=-\infty}^{\infty} i^{\ell} J_{\ell}(z)\cos\ell\vartheta,$$

see [Lou06].

#### 16. **2-D** Dirac delta on sphere

$$\delta(L_1, L_2) = \sum_{\ell, m} Y_{\ell}^m(L_1) Y_{\ell}^m(L_2) = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) P_{\ell}(L_1 \cdot L_2),$$

#### 1-D Dirac delta

$$\delta(x - y) = \frac{1}{2} \sum_{\ell} (2\ell + 1) P_{\ell}(x) P_{\ell}(y),$$

see [Lou06].

### 17. Rotational invariant functions. The function

$$I_{\ell}(L_{1}, L_{2}) = \frac{4\pi}{2\ell + 1} \sum_{k=-\ell}^{\ell} (-1)^{k} Y_{\ell}^{k}(L_{1}) Y_{\ell}^{-k}(L_{2})$$

$$= \frac{4\pi}{2\ell + 1} \sum_{k=-\ell}^{\ell} Y_{\ell}^{k}(L_{1}) Y_{\ell}^{k*}(L_{2})$$
(31)

is rotational invariant. This expression is valid when we apply rotation  $g_{L_1}$ , where  $g_{L_1}L_1=N$ 

$$I_{\ell}(L_1, L_2) = \frac{4\pi}{2\ell + 1} \frac{2\ell + 1}{4\pi} \sum_{k=-\ell}^{\ell} \delta_{k,0} Y_{\ell}^{k} (g_{L_1} L_2)$$
$$= Y_{\ell}^{0} (g_{L_1} L_2)$$
$$= P_{\ell} (\cos L_1 \cdot L_2).$$

It follows from the addition formula (42) as well. The function

$$I_{\ell_1\ell_2\ell_3}\left(L_1, L_2, L_3\right) = \frac{\left(4\pi\right)^{3/2}}{\sqrt{\prod (2\ell_j + 1)}} \sum_{m_{1,2}} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} Y_{\ell_1}^{m_1}\left(L_1\right) Y_{\ell_2}^{m_2}\left(L_2\right) Y_{\ell_3}^{m_3}\left(L_3\right) \quad (32)$$

of three location is rotational invariant, see [Lou06], pp.14. We repeat the notation  $\begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ . In particular we have

$$I_{\ell_{1:3}}(L_{1:3}) = I_{\ell_{1:3}}(N, g_{L_1}L_2, g_{L_1}L_3).$$

18. Wigner D-matrix Let  $\Lambda(g) Y_{\ell}^{m}(L) = Y_{\ell}^{m}(g^{-1}L)$ ,

$$\Lambda(g) Y_{\ell}^{m}(L) = \sum_{k=-\ell}^{\ell} D_{k,m}^{(\ell)}(g) Y_{\ell}^{k}(L)$$
(33)

if  $\ell$  is fixed  $D_{m,k}^{(\ell)}\left(g\right)$  is unitary

$$\sum_{k=-\ell}^{\ell} D_{m_1,k}^{(\ell)}(g) D_{m_2,k}^{(\ell)*}(g) = \delta_{m_1,m_2},$$

$$\sum_{k=-\ell}^{\ell} D_{k,m_1}^{(\ell)}(g) D_{k,m_2}^{(\ell)*}(g) = \delta_{m_1,m_2},$$

$$\sum_{k=-\ell}^{\ell} D_{k,m_1}^{(\ell)*}(g) D_{k,m_2}^{(\ell)}(g) = \delta_{m_1,m_2}.$$

We introduce the notation  $D^{(\ell)} = \left[D_{m,k}^{(\ell)}\right]$ , for fixed rotation g. Thus  $D^{(\ell)}$  denotes a unitary matrix of order  $2\ell+1$ , and it follows  $D^{(\ell)}\left[D^{(\ell)}\right]^{-1} = D^{(\ell)}D^{(\ell)*}$ ,  $\det D^{(\ell)} = 1$  (unimodular).  $D_{m,k}^{(\ell)}\left(g\right)$  is given by the clear formula

$$D_{m,k}^{(\ell)}(\varphi,\vartheta,\gamma) = \exp(-im\varphi) d_{m,k}^{(\ell)}(\vartheta) \exp(-ik\gamma)$$
(34)

where  $d_{m,k}^{(\ell)}$  is the Wigner (small) d-matrix it is real and

$$\begin{split} d_{m,k}^{(\ell)}\left(-\vartheta\right) &= \left(-1\right)^{m-k} d_{m,k}^{(\ell)}\left(\vartheta\right) \\ &= d_{k,m}^{(\ell)}\left(\vartheta\right) \\ &= \left(-1\right)^{k-m} d_{-k,-m}^{(\ell)}\left(\vartheta\right) \\ &= d_{m-k}^{(\ell)}\left(\vartheta\right) \end{split}$$

see [VMK88], pp79 for details for instance

$$\begin{split} \left[D_{m,k}^{(\ell)}\left(\varphi,\vartheta,\gamma\right)\right]^* &= \left[\overline{D_{k,m}^{(\ell)}}\left(\varphi,\vartheta,\gamma\right)\right] = \left[(-1)^{m-k}\,\overline{D_{m,k}^{(\ell)}}\left(\gamma,\vartheta,\varphi\right)\right] \\ &= \left[(-1)^{m-k}\,D_{-m,-k}^{(\ell)}\left(\gamma,\vartheta,\varphi\right)\right] \\ &= \left[D_{m,k}^{(\ell)}\left(-\gamma,-\vartheta,-\varphi\right)\right], \end{split}$$

$$\begin{split} D_{k,m}^{(\ell)}\left(\varphi,\vartheta,\gamma\right) &= (-1)^{m-k} \, D_{m,k}^{(\ell)}\left(\gamma,\vartheta,\varphi\right) \\ &= D_{m,k}^{(\ell)}\left(\gamma,-\vartheta,\varphi\right) \end{split}$$

note here that if the Euler angles  $(\varphi, \vartheta, \gamma)$  corresponds to g then  $(-\gamma, -\vartheta, -\varphi)$  corresponds to  $g^{-1}$ , hence

$$D_{m,k}^{(\ell)}(g) = (-1)^{m-k} D_{-m,-k}^{(\ell)*}(g)$$

$$D_{m,k}^{(\ell)*}(g) = D_{k,m}^{(\ell)}(g^{-1})$$
(35)

also

$$D_{m,k}^{(\ell)}(g) = D_{\ell-m+1,\ell-k+1}^{(\ell)}(g)$$

Let g the successive application of  $g_1$  and  $g_2$  then

$$D_{m,k}^{(\ell)}(g) = \sum_{j=-\ell}^{\ell} D_{m,j}^{(\ell)}(g_1) D_{j,k}^{(\ell)}(g_2).$$

19. Condon and Shortley phase convention, [Edm57], (4.3.3)

$$Y_{\ell}^{m}(\vartheta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}} D_{0,-m}^{(\ell)}(\gamma,\vartheta,\varphi)$$

$$= (-1)^{m} D_{0,m}^{(\ell)*}(\gamma,\vartheta,\varphi)$$

$$= (-1)^{m} \sqrt{\frac{2\ell+1}{4\pi}} D_{-m,0}^{(\ell)}(\varphi,\vartheta,\gamma)$$

$$= \sqrt{\frac{2\ell+1}{4\pi}} D_{m,0}^{(\ell)*}(\varphi,\vartheta,\gamma),$$
(36)

where  $\gamma$  is arbitrary angle. This form is referred to as passive convention as well, see [MP87]. It also follows

$$\begin{split} Y_{\ell}^{m*}\left(\vartheta,\varphi\right) &= Y_{\ell}^{m}\left(\vartheta,-\varphi\right) \\ &= \left(-1\right)^{m} Y_{\ell}^{-m}\left(\vartheta,\varphi\right), \\ Y_{\ell}^{-m}\left(\vartheta,\varphi\right) &= \left(-1\right)^{m} e^{-i2m\varphi} Y_{\ell}^{m}\left(\vartheta,\varphi\right). \end{split}$$

[Wor], (52) in terms of Wigner 3j-symbols, [Edm57] (4.3.2)

20. Singly coupled form, Clebsch-Gordan series:

$$D_{m_1,k_1}^{(\ell_1)} D_{m_2,k_2}^{(\ell_2)} = \sum_{\ell,m,k} (2\ell+1) \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & m \end{pmatrix} D_{m,k}^{(\ell)*} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & k \end{pmatrix}$$

$$= \sum_{\ell,m,k} C_{\ell_1,m_1;\ell_2,m_2}^{\ell,m} C_{\ell_1,k_1;\ell_2,k_2}^{\ell,k} D_{m,k}^{(\ell)},$$
(37)

$$D_{m_1,k_1}^{(\ell_1)}D_{m_2,k_2}^{(\ell_2)} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & m \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & k \end{pmatrix} (-1)^{m-k} (2\ell+1) D_{m_1+m_2,k_1+k_2}^{(\ell)}$$

 $-m = m_1 + m_2, -k = k_1 + k_2,$  [Edm57], (4.3.4). Double coupled form

$$\sum_{m_1, m_2, k_1, k_2} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & m \end{pmatrix} D_{m_1, k_1}^{(\ell_1)} D_{m_2, k_2}^{(\ell_2)} \begin{pmatrix} \ell_{1:2} & j \\ k_{1:2} & k \end{pmatrix} = \frac{\delta_{\ell, j}}{2\ell + 1} D_{m, k_1}^{(\ell)*}$$

[Edm57], (4.3.3)

$$\sum_{m_1, m_2, m_3} D_{m_1, k_1}^{(\ell_1)} D_{m_2, k_2}^{(\ell_2)} D_{m_3, k_3}^{(\ell_3)} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix}, \tag{38}$$

The integrals

$$\int_{SO(3)} D_{m,k}^{(\ell)} dg = \delta_{\ell,0} \delta_{m,0} \delta_{k,0} \tag{39}$$

$$\mathcal{G}_{k_1,k_2;m_1,m_2}^{\ell_1,\ell_2} = \int_{SO(3)} D_{m_1,k_1}^{(\ell_1)*} D_{m_2,k_2}^{(\ell_2)} dg = \delta_{\ell_1,\ell_2} \delta_{m_1,m_2} \delta_{k_1,k_2} \frac{1}{2\ell_1 + 1}$$
(40)

$$\mathcal{G}_{k_1,k_2,k_3;m_1,m_2,m_3}^{\ell_1,\ell_2,\ell_3} = \int_{SO(3)} D_{m_1,k_1}^{(\ell_1)} D_{m_2,k_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)} dg = \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix}$$
(41)

$$\int_{SO(3)} D_{m_1,k_1}^{(\ell_1)*} D_{m_2,k_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)} dg = \frac{1}{2\ell_3 + 1} C_{\ell_1,m_1;\ell_2,m_2}^{\ell_3,m_3} C_{\ell_1,k_1;\ell_2,k_2}^{\ell_3,k_3}$$

where the Haar measure is  $dg = \sin \theta d\theta d\varphi d\gamma / 8\pi^2$ , [VMK88], 4.11.1, [Edm57], (4.6.2)

21. Addition formula [GR00], 8.814, [EMOT81] vol. 2, 11.4(8) pp. 236, [Ml66].

$$\sum_{m=-\ell}^{\ell} Y_{\ell}^{m*} (L_1) Y_{\ell}^{m} (L_2) = \frac{2\ell+1}{4\pi} P_{\ell} (\cos \theta)$$
 (42)

where  $\cos \vartheta = L_1 \cdot L_2$ . For general dimension d [Yad83] p.72 provides an addition formula in terms of Gegenbauer polynomials  $C_\ell^{\nu}$ .

# C Proofs, Bispectrum

We use the following notations  $Z_{\ell_{1:3}}^{m_{1:3}} = \left(Z_{\ell_{1}}^{m_{1}}, Z_{\ell_{2}}^{m_{2}}, Z_{\ell_{3}}^{m_{3}}\right), \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = \begin{pmatrix} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & m_{3} \end{pmatrix}, B_{3}\left(\ell_{1:3}\right) = B_{3}\left(\ell_{1}, \ell_{2}, \ell_{3}\right).$ 

Proof of Lemma. 13 Let

$$\operatorname{Cum}_{3}\left(Z_{\ell_{1:3}}^{m_{1:3}}\right) = \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} B_{3}\left(\ell_{1:3}\right),$$

then

$$\operatorname{Cum}_{3}\left(\mathcal{Z}_{\ell_{1:3}}^{k_{1:3}}\right) = \sum_{m_{1},m_{2},m_{3}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{m_{3},k_{3}}^{(\ell_{3})} \operatorname{Cum}_{3}\left(Z_{\ell_{1:3}}^{m_{1:3}}\right)$$

$$= \sum_{m_{1:3}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{m_{3},k_{3}}^{(\ell_{3})} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} B_{3}\left(\ell_{1:3}\right)$$

$$= \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix} B_{3}\left(\ell_{1:3}\right)$$

$$= \operatorname{Cum}_{3}\left(Z_{\ell_{1}}^{k_{1}}, Z_{\ell_{2}}^{k_{2}}, Z_{\ell_{3}}^{k_{3}}\right)$$

$$(43)$$

where  $\mathcal{Z}_{\ell_{1:3}}^{k_{1:3}} = \left(\mathcal{Z}_{\ell_{1}}^{k_{1}}, \mathcal{Z}_{\ell_{2}}^{k_{2}}, \mathcal{Z}_{\ell_{3}}^{k_{3}}\right)$  see (38). Hence the assumption of isotropy is fulfilled. If (8) is not assumed then under the assumption of isotropy we have  $\operatorname{Cum}_{3}\left(\mathcal{Z}_{\ell_{1:3}}^{k_{1:3}}\right) = \operatorname{Cum}_{3}\left(\mathcal{Z}_{\ell_{1:3}}^{k_{1:3}}\right)$ , and integrate both sides of (43) according to the Haar measure and obtain

$$\operatorname{Cum}_{3}\left(Z_{\ell_{1:3}}^{k_{1:3}}\right) = {\ell_{1:3} \choose k_{1:3}} \sum_{m_{1:3}} {\ell_{1:3} \choose m_{1:3}} \operatorname{Cum}_{3}\left(Z_{\ell_{1:3}}^{m_{1:3}}\right)$$
$$= {\ell_{1:3} \choose k_{1:3}} B_{3}\left(\ell_{1:3}\right)$$

see (41), where

$$B_3(\ell_{1:3}) = \sum_{m_{1:3}} {\ell_{1:3} \choose m_{1:3}} \operatorname{Cum}_3(Z_{\ell_{1:3}}^{m_{1:3}}).$$

Hence (8) is a necessary and sufficient assumption for the third order isotropy. In this case the bispectrum is a linear combination of the cumulants fo the angular projections by the probability amplitude of coupling three angular momenta  $\ell_{1:3}$ .

**Proof of (9).** Consider the expression

$$\begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix} B_3 \left( \ell_{1:3} \right) = \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix} \sum_{m_{1:3}} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} \operatorname{Cum}_3 \left( Z_{\ell_{1:3}}^{m_{1:3}} \right).$$

Any permutation of  $\ell_{1:3}$  implies the same sign of the Wigner coefficients hence it does not change the value.

Proof: 3-product of spherical harmonics is rotation invariant. Indeed

$$I_{\ell_{1:3}}(gL_{1:3}) = \sum_{m_{1:3}} {\ell_{1:3} \choose m_{1:3}} \sum_{k_{1:3}} D_{k_1,m_1}^{(\ell_1)} D_{k_2,m_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)} Y_{\ell_1}^{k_1} (L_1) Y_{\ell_2}^{k_2} (L_2) Y_{\ell_3}^{k_3} (L_3)$$

$$= \sum_{k_{1:2}} {\ell_{1:3} \choose k_{1:3}} Y_{\ell_1}^{k_1} (L_1) Y_{\ell_2}^{k_2} (L_2) Y_{\ell_3}^{k_3} (L_3).$$

see (38). ■

# D Proofs, Trispectrum

Repeat the notations  $\ell_{1:4} = (\ell_1, \ell_2, \ell_3, \ell_4)$ ,  $\operatorname{Cum}_4(X(L_1), X(L_2), X(L_3), X(L_4)) = \operatorname{Cum}_4(X(L_{1:4}))$ ,  $T_4(\ell_1, \ell_2, \ell_3, \ell_4 | \ell^1) = T_4(\ell_{1:4} | \ell^1)$ .

Lemma 27

$$\sum_{m_{1:4}} D_{k_1,m_1}^{(\ell_1)} D_{k_2,m_2}^{(\ell_2)} D_{k_3,m_3}^{(\ell_3)} D_{k_4,m_4}^{(\ell_4)} (-1)^m \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ m & m_{3:4} \end{pmatrix} = (-1)^k \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell \\ k_{3:4} & k \end{pmatrix}.$$

**Proof.** Indeed, from

$$D_{k_3,m_3}^{(\ell_3)}D_{k_4,m_4}^{(\ell_4)} = \sum_{\ell^2 \ k^2 \ m^2} \left(2\ell^2 + 1\right) \begin{pmatrix} \ell_{3:4} & \ell^2 \\ m_{3:4} & m^2 \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell^2 \\ k_{3:4} & k^2 \end{pmatrix} D_{k^2,m^2}^{(\ell^2)*},$$

and from  $m_1 + m_2 + m_3 + m_4 = 0$ , it follows

$$\begin{split} &\sum_{m_{1:4}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{k_{3},m_{3}}^{(\ell_{3})} D_{k_{4},m_{4}}^{(\ell_{4})} \left(-1\right)^{m} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ m & m_{3:4} \end{pmatrix} \\ &= \sum_{\ell^{2},k^{2},m^{2}} \sum_{m_{1:4}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{k^{2},m^{2}}^{(\ell^{2})*} \left(2\ell^{2}+1\right) \\ &\times \left(-1\right)^{m} \begin{pmatrix} \ell_{3:4} & \ell^{2} \\ m_{3:4} & m^{2} \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell^{2} \\ k_{3:4} & k^{2} \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ m & m_{3:4} \end{pmatrix} \\ &= \sum_{\ell^{2},k^{2},m^{2}} \sum_{m_{3:4}} \begin{pmatrix} \ell_{3:4} & \ell^{2} \\ m_{3:4} & -m^{2} \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ m & m_{3:4} \end{pmatrix} \left(2\ell^{2}+1\right) \\ &\times \left(-1\right)^{m} \sum_{m_{1:2}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{k^{2},m^{2}}^{(\ell^{2})} \left(-1\right)^{m^{2}-k^{2}} \begin{pmatrix} \ell_{3:4} & \ell^{2} \\ k_{3:4} & -k^{2} \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \\ &= \delta_{m,-m^{2}} \sum_{\ell^{2},k^{2},m^{2}} \sum_{m_{1:2}} \delta_{\ell,\ell^{2}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{-k^{2},m^{2}}^{(\ell^{2})} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m^{2} \end{pmatrix} \left(-1\right)^{-k^{2}} \begin{pmatrix} \ell_{3:4} & \ell \\ k_{3:4} & -k^{2} \end{pmatrix} \\ &= \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & k^{2} \end{pmatrix} \left(-1\right)^{-k^{2}} \begin{pmatrix} \ell_{3:4} & \ell \\ k_{3:4} & -k^{2} \end{pmatrix}, \end{split}$$

since 3j-symbols are orthogonal

$$(2\ell+1)\sum_{m_{1:2}} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell^1 \\ m_{1:2} & m^1 \end{pmatrix} = \delta_{\ell,\ell^1} \delta_{m,-m^1},$$

see (29) and (38) is valid.

#### Lemma 28

$$\int_{SO(3)} D_{k_1,m_1}^{(\ell_1)} D_{k_2,m_2}^{(\ell_2)} D_{k_3,m_3}^{(\ell_3)} D_{k_4,m_4}^{(\ell_4)} dg$$

$$= \sum_{\ell} (2\ell+1) (-1)^{k-m} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell \\ k_{3:4} & k \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell \\ m_{3:4} & m \end{pmatrix}$$

**Proof.** Here we apply the formula (37) for the single coupled D-matrices then the integral of triple product of D-matrices (41) and get

$$\begin{split} &\int_{SO(3)} D_{k_1,m_1}^{(\ell_1)} D_{k_2,m_2}^{(\ell_2)} D_{k_3,m_3}^{(\ell_3)} D_{k_4,m_4}^{(\ell_4)} dg \\ &= \sum_{\ell^1,k^1,m^1} \left( 2\ell^1 + 1 \right) (-1)^{k^1 - m^1} \begin{pmatrix} \ell_{3:4} & \ell^1 \\ k_{3:4} & k^1 \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell^1 \\ m_{3:4} & m^1 \end{pmatrix} \int_{SO(3)} D_{k_1,m_1}^{(\ell_1)} D_{k_2,m_2}^{(\ell_2)} D_{-k^1,-m^1}^{(\ell^1)} dg \\ &= \sum_{\ell^1,k^1,m^1} \begin{pmatrix} \ell_{1:2} & \ell^1 \\ k_{1:2} & -k^1 \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell^1 \\ k_{3:4} & k^1 \end{pmatrix} \\ &\times \left( 2\ell^1 + 1 \right) (-1)^{k^1 - m^1} \begin{pmatrix} \ell_{1:2} & \ell^1 \\ m_{1:2} & -m^1 \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell^1 \\ m_{3:4} & m^1 \end{pmatrix}. \end{split}$$

**Lemma 29** If  $\ell_1 + \ell_2 + \ell_3 + \ell_4$  is even then

$$\sum_{\ell} (2\ell+1) (-1)^{k-m} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ k & k_{3:4} \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ m & m_{3:4} \end{pmatrix}$$

is symmetric in  $\ell_{1:4}$ .

**Proof.** Start with

$$\begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ k & k_{3:4} \end{pmatrix}$$

$$= \sum_{\ell_0} (2\ell_0 + 1) (-1)^{\sum + \ell_0 - \ell - k_1 - k_3} \begin{pmatrix} \ell_2 & \ell_3 & \ell_0 \\ k_2 & k_3 & -k_0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_0 & \ell_4 \\ k_1 & k_0 & k_4 \end{pmatrix} \begin{cases} \ell_1 & \ell_2 & \ell \\ \ell_3 & \ell_4 & \ell_0 \end{cases}$$

where  $\Sigma = \ell_1 + \ell_2 + \ell_3 + \ell_4$ . Now we evaluate the product

$$\sum_{\ell_0^1} \left( 2\ell_0^1 + 1 \right) (-1)^{\Sigma + \ell_0^1 - \ell - k_1 - k_3} \begin{pmatrix} \ell_2 & \ell_3 & \ell_0^1 \\ k_2 & k_3 & -k_0^1 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_0^1 & \ell_4 \\ k_1 & k_0^1 & k_4 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell \\ \ell_3 & \ell_4 & \ell_0^1 \end{Bmatrix} \\
\times \sum_{\ell_2^2} \left( 2\ell_0^2 + 1 \right) (-1)^{\Sigma + \ell_0^2 - \ell - m_1 - m_3} \begin{pmatrix} \ell_2 & \ell_3 & \ell_0^2 \\ m_2 & m_3 & -m_0^2 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_0^2 & \ell_4 \\ m_1 & m_0^2 & m_4 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell \\ \ell_3 & \ell_4 & \ell_0^2 \end{Bmatrix}.$$

The values

$$\sqrt{\left(2\ell+1\right)\left(2\ell_0^1+1\right)} \begin{cases} \ell_1 & \ell_2 & \ell \\ \ell_3 & \ell_4 & \ell_0^1 \end{cases}$$

form a real orthogonal matrix, see [Edm57] (6.2.10), rows and columns being labelled by  $\ell$  and  $\ell_0^1$ . Hence if we sum it up by  $\ell$ , the result is

$$\sum_{\ell_0^1} \left( 2\ell_0^1 + 1 \right) (-1)^{-k_0^1} \begin{pmatrix} \ell_2 & \ell_3 & \ell_0^1 \\ k_2 & k_3 & -k_0^1 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_0^1 & \ell_4 \\ k_1 & k_0^1 & k_4 \end{pmatrix} \times (-1)^{-m_0^1} \begin{pmatrix} \ell_2 & \ell_3 & \ell_0^1 \\ m_2 & m_3 & -m_0^1 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_0^1 & \ell_4 \\ m_1 & m_0^1 & m_4 \end{pmatrix},$$

q.e.d. ■

**Proof of Lemma 16.** We have

$$\operatorname{Cum}_{4}\left(\mathcal{Z}_{\ell_{1:4}}^{m_{1:4}}\right) = \sum_{k_{1:4}} D_{m_{1},k_{1}}^{(\ell_{1})} D_{m_{2},k_{2}}^{(\ell_{2})} D_{m_{3},k_{3}}^{(\ell_{3})} D_{m_{4},k_{4}}^{(\ell_{4})} \operatorname{Cum}_{4}\left(Z_{\ell_{1:4}}^{k_{1:4}}\right)$$

Under the assumption of isotropy  $\operatorname{Cum}_4\left(\mathcal{Z}^{m_{1:4}}_{\ell_{1:4}}\right) = \operatorname{Cum}_4\left(\mathcal{Z}^{m_{1:4}}_{\ell_{1:4}}\right)$ , now integrate both sides by the Haar measure, see Lemma 30 for p=4, and obtain

$$\operatorname{Cum}_{4}\left(Z_{\ell_{1:4}}^{m_{1:4}}\right) = \sum_{k_{1:4}} \sum_{\ell^{1}, k^{1}, m^{1}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ m_{1:2} & -m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3:4} \\ m^{1} & m_{3:4} \end{pmatrix} (-1)^{m^{1}-k^{1}} \\ \times \left(2\ell^{1}+1\right) \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ k_{1:2} & -k^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3:4} \\ k^{1} & k_{3:4} \end{pmatrix} \operatorname{Cum}_{4} \left(Z_{\ell_{1:4}}^{k_{1:4}}\right)$$

Note that each term according to summation  $k_{1:4}$  is *symmetric* in  $\ell_{1:4}$ , see Lemma 29 above. Now, define

$$T_4\left(\ell_{1:4}|\ell^1\right) = \sum_{k^1, k_{1:4}} (-1)^{k^1} \sqrt{2\ell^1 + 1} \begin{pmatrix} \ell_{1:2} & \ell^1 \\ k_{1:2} & -k^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_{3:4} \\ k^1 & k_{3:4} \end{pmatrix} \operatorname{Cum}_4\left(Z_{\ell_{1:4}}^{k_{1:4}}\right),$$

with this notation we have the cumulant in the form

$$\operatorname{Cum}_{4}\left(Z_{\ell_{1:4}}^{m_{1:4}}\right) = \sum_{\ell^{1} \ m^{1}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ m_{1:2} & -m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3:4} \\ m^{1} & m_{3:4} \end{pmatrix} (-1)^{m^{1}} \sqrt{2\ell^{1} + 1} T_{4} \left(\ell_{1:4} | \ell^{1}\right).$$

If instead of isotropy (11) is assumed then

$$\begin{aligned} \operatorname{Cum}_{4}\left(\mathcal{Z}_{\ell_{1:4}}^{m_{1:4}}\right) &= \sum_{k_{1:4}} D_{m_{1},k_{1}}^{(\ell_{1})} D_{m_{2},k_{2}}^{(\ell_{2})} D_{m_{3},k_{3}}^{(\ell_{4})} D_{m_{4},k_{4}}^{(\ell_{4})} \operatorname{Cum}_{4}\left(Z_{\ell_{1:4}}^{k_{1:4}}\right) \\ &= \sum_{k_{1:4}} D_{m_{1},k_{1}}^{(\ell_{1})} D_{m_{2},k_{2}}^{(\ell_{2})} D_{m_{3},k_{3}}^{(\ell_{3})} D_{m_{4},k_{4}}^{(\ell_{4})} \\ &\times \sum_{\ell} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ k & k_{3:4} \end{pmatrix} (-1)^{k} \sqrt{2\ell + 1} T_{4} \left(\ell_{1:4} | \ell\right) \\ &= \sum_{\ell} T_{4} \left(\ell_{1:4} | \ell\right) \sqrt{2\ell + 1} \\ &\times \sum_{k_{1:4}} D_{m_{1},k_{1}}^{(\ell_{1})} D_{m_{2},k_{2}}^{(\ell_{2})} D_{m_{3},k_{3}}^{(\ell_{3})} D_{m_{4},k_{4}}^{(\ell_{4})} \left(-1\right)^{k} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ k & k_{3:4} \end{pmatrix}, \end{aligned}$$

the Lemma 27 can be applied and we get

$$\operatorname{Cum}_{4}\left(\mathcal{Z}_{\ell_{1:4}}^{m_{1:4}}\right) = \sum_{m,\ell} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \begin{pmatrix} \ell_{3:4} & \ell \\ m_{3:4} & m \end{pmatrix} (-1)^{m} \sqrt{2\ell + 1} T_{4} \left(\ell_{1:4} | \ell\right)$$

$$= \operatorname{Cum}_{4}\left(Z_{\ell_{1:4}}^{m_{1:4}}\right).$$

Particular cases

$$\operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(L\right), u_{\ell_{2}}\left(L\right), u_{\ell_{3}}\left(L\right), u_{\ell_{4}}\left(L\right)\right) = \sum_{m, \ell} T_{4}\left(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} | \ell\right) \sqrt{2\ell + 1} \sum_{m_{1:4}} \prod_{j=1}^{4} Y_{\ell_{j}}^{m_{j}}\left(L_{j}\right) \times \begin{pmatrix} \ell_{1} & \ell_{2} & \ell\\ m_{1} & m_{2} & -m \end{pmatrix} \begin{pmatrix} \ell & \ell_{3} & \ell_{4}\\ m & m_{3} & 0 \end{pmatrix},$$

$$\begin{aligned} \operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(L_{1}\right),u_{\ell_{2}}\left(L_{2}\right),u_{\ell_{3}}\left(L\right),u_{\ell_{4}}\left(L\right)\right) &= \operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(g_{LL_{2}}L_{1}\right),u_{\ell_{2}}\left(g_{LL_{2}}L_{2}\right),u_{\ell_{3}}\left(N\right),u_{\ell_{4}}\left(N\right)\right) \\ &= \sqrt{\prod_{j=3:4}}\frac{2\ell_{j}+1}{4\pi}\sum_{\ell}T_{4}\left(\ell_{1},\ell_{2},\ell_{3},\ell_{4}|\ell\right)\begin{pmatrix}\ell_{3}&\ell_{4}&\ell\\0&0&0\end{pmatrix}\sqrt{2\ell+1} \\ &\times\sum_{m_{1},m_{2}}Y_{\ell_{1}}^{m_{1}}\left(g_{LL_{2}}L_{1}\right)Y_{\ell_{2}}^{m_{2}}\left(g_{LL_{2}}L_{2}\right)\begin{pmatrix}\ell_{1:2}&\ell\\m_{1:2}&0\end{pmatrix} \\ &= \frac{\sqrt{2}}{4\pi}\sqrt{\prod_{j=3:4}}\frac{2\ell_{j}+1}{4\pi}\sum_{\ell}\begin{pmatrix}\ell_{3}&\ell_{4}&\ell\\0&0&0\end{pmatrix}T_{4}\left(\ell_{1},\ell_{2},\ell_{3},\ell_{4}|\ell\right)\mathcal{I}_{\ell_{1},\ell_{2},\ell}\left(\vartheta_{2},\varphi_{2},\vartheta_{1}\right), \end{aligned}$$

$$\operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(L_{1}\right), u_{\ell_{2}}\left(L\right), u_{\ell_{3}}\left(L\right), u_{\ell_{4}}\left(L\right)\right) = \operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(g_{LL_{1}}L_{1}\right), u_{\ell_{2}}\left(N\right), u_{\ell_{3}}\left(N\right), u_{\ell_{4}}\left(N\right)\right)$$

$$= \sqrt{\prod_{j=2:4} \frac{2\ell_{j}+1}{4\pi}} \sum_{\ell} T_{4}\left(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} \middle| \ell\right) \sqrt{2\ell+1} Y_{\ell_{1}}^{0}\left(g_{LL_{1}}L_{1}\right) \begin{pmatrix} \ell_{1} & \ell_{2} & \ell\\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell_{3} & \ell_{4}\\ 0 & 0 & 0 \end{pmatrix}$$

$$= \sqrt{\prod_{j=1:4} \frac{2\ell_{j}+1}{4\pi}} \sum_{\ell} T_{4}\left(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} \middle| \ell\right) \sqrt{2\ell+1} P_{\ell_{1}}\left(L \cdot L_{1}\right) \begin{pmatrix} \ell_{1} & \ell_{2} & \ell\\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell_{3} & \ell_{4}\\ 0 & 0 & 0 \end{pmatrix},$$

$$\operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(L\right),u_{\ell_{2}}\left(L\right),u_{\ell_{3}}\left(L\right),u_{\ell_{4}}\left(L\right)\right) = \operatorname{Cum}_{4}\left(u_{\ell_{1}}\left(N\right),u_{\ell_{2}}\left(N\right),u_{\ell_{3}}\left(N\right),u_{\ell_{4}}\left(N\right)\right)$$

$$= \sqrt{\prod_{j=1:4} \frac{2\ell_{i}+1}{4\pi}} \sum_{\ell} T_{4}\left(\ell_{1},\ell_{2},\ell_{3},\ell_{4}|\ell\right) \sqrt{2\ell+1} \begin{pmatrix} \ell_{1} & \ell_{2} & \ell\\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell_{3} & \ell_{4}\\ 0 & 0 & 0 \end{pmatrix}.$$

# E Proofs, Polyspectrum

Notation:  $\mathcal{G}_{k_{1:p},m_{1:p}}^{\ell_{1:p}} = \mathcal{G}_{k_{1},k_{2},\dots,k_{p},m_{1},m_{2},\dots,m_{p}}^{\ell_{1},\ell_{2},\dots,\ell_{3p}}$ .

**Lemma 30** Define  $\ell^0 = \ell_1$ ,  $\ell^{p-2} = \ell_p$ ,  $k^0 = -k_1$ ,  $k^{p-2} = k_p$ ,  $m^0 = -m_1$ ,  $m^{p-2} = m_p$ , then for p > 3,

$$\begin{split} \mathcal{G}_{k_{1:p},m_{1:p}}^{\ell_{1:p}} &= \int_{SO(3)} \prod_{a=1}^{p} D_{k_{a},m_{a}}^{(\ell_{a})} dg = \sum_{\ell^{a},m^{a},k^{a}} (-1)^{\sum \left(m^{1:p-3}-k^{1:p-3}\right)} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -k^{a} & k_{a+2} & k^{a+1} \end{pmatrix} \\ & \times \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix} \prod_{a=1}^{p-3} \left(2\ell^{a}+1\right), \end{split}$$

**Proof.** The fourth order key formula is emphasized here for understanding the induction

$$\begin{split} &\mathcal{G}_{k_{1:5};m_{1:5}}^{\ell_{1:5}} = \int_{SO(3)} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{k_{3},m_{3}}^{(\ell_{3})} D_{k_{4},m_{4}}^{(\ell_{5})} D_{k_{5},m_{5}}^{(\ell_{5})} dg \\ &= \sum_{\ell^{2},k^{2},m^{2}} \int_{SO(3)} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{k_{3},m_{3}}^{(\ell_{3})} D_{-k^{2},-m^{2}}^{(\ell^{2})} dg \\ &\times (-1)^{m^{2}-k^{2}} \left(2\ell^{2}+1\right) \begin{pmatrix} \ell_{4:5} & \ell^{2} \\ k_{4:5} & k^{2} \end{pmatrix} \begin{pmatrix} \ell_{4:5} & \ell^{2} \\ m_{4:5} & m^{2} \end{pmatrix} \\ &= \sum_{\ell^{1:2},k^{1:2},m^{1:2}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ k_{1:2} & k^{1} \end{pmatrix} \begin{pmatrix} \ell_{3} & \ell^{2} & \ell^{1} \\ k_{3} & k^{2} & -k^{1} \end{pmatrix} \begin{pmatrix} \ell_{4:5} & \ell^{2} \\ k_{4:5} & -k^{2} \end{pmatrix} (-1)^{m^{1}-k^{1}} \\ &\times \left(2\ell^{1}+1\right) \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ m_{1:2} & m^{1} \end{pmatrix} \begin{pmatrix} \ell_{3} & \ell^{2} & \ell^{1} \\ m_{3} & m^{2} & -m^{1} \end{pmatrix} \begin{pmatrix} \ell_{4:5} & \ell^{2} \\ m_{4:5} & -m^{2} \end{pmatrix} \\ &\times (-1)^{m^{2}-k^{2}} \left(2\ell^{2}+1\right) \\ &= \sum_{\ell^{1:2},k^{1:2},m^{1:2}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ k_{1:2} & k^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell^{2} \\ -k^{1} & k_{3} & k^{2} \end{pmatrix} \begin{pmatrix} \ell^{2} & \ell_{4:5} \\ -k^{2} & k_{4:5} \end{pmatrix} (-1)^{m^{1}-k^{1}} \left(2\ell^{1}+1\right) \\ &\times \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ m_{1:2} & m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell^{2} \\ -m^{1} & m_{3} & m^{2} \end{pmatrix} \begin{pmatrix} \ell^{2} & \ell_{4:5} \\ -m^{2} & m_{4:5} \end{pmatrix} \times (-1)^{m^{2}-k^{2}} \left(2\ell^{2}+1\right). \end{split}$$

Define  $\ell^{p-2} = \ell_p, \ k^{p-2} = k_p, \ m^{p-2} = m_p$ 

$$\int_{SO(3)} \prod_{a=1}^{p} D_{k_a, m_a}^{(\ell_a)} dg = (-1)^{m_p} \sum_{\ell^a, m^a, k^a} \prod_{a=0}^{p-3} C_{\ell^a, k^a; \ell_{a+2}, k_{a+2}}^{\ell^{a+1}, k^{a+1}} C_{\ell^a, m^a; \ell_{a+2}, m_{a+2}}^{\ell^{a+1}, m^{a+1}}$$

$$\begin{split} \int_{SO(3)} \prod_{a=1}^{p} D_{k_{a},m_{a}}^{(\ell_{a})} dg &= \sum_{\ell^{a},m^{a},k^{a}} (-1)^{\sum \left(m^{1:p-3}-k^{1:p-3}\right)} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ k_{1:2} & k^{1} \end{pmatrix} \prod_{a=1}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -k^{a} & k_{a+2} & k^{a+1} \end{pmatrix} \\ & \times \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ m_{1:2} & m^{1} \end{pmatrix} \prod_{a=1}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix} (2\ell^{a} + 1) \end{split}$$

**Lemma 31** Define  $\ell^0 = \ell_1$ ,  $\ell^{p-2} = \ell_p$ ,  $k^0 = -k_1$ ,  $k^{p-2} = k_p$ ,  $m^0 = -m_1$ ,  $m^{p-2} = m_p$ , then for p < 3 and for any  $\ell^{1:p-3} = (\ell^1, \ell^2, \dots, \ell^{p-3})$  we have

$$\mathcal{H}_{k_{1:p},m_{1:p}}^{\ell_{1:p}}\left(\ell^{1:p-3}\right) = \sum_{m_{1:p}} \prod_{a=1}^{p} D_{k_{a},m_{a}}^{(\ell_{a})} \left(-1\right)^{\sum m^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$= (-1)^{\sum k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -k^{a} & k_{a+2} & k^{a+1} \end{pmatrix}.$$

**Proof.** Induction from p=4 to p=5 will show the general tratment. We prove that  $\mathcal{H}_{k_1:5,m_1:5}^{\ell_1:5}\left(\ell^{1:2}\right)$ 

$$\mathcal{H}_{k_{1:5},m_{1:5}}^{\ell_{1:5}}\left(\ell^{1:2}\right) \\ = \sum_{m_{1:5}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{k_{3},m_{3}}^{(\ell_{3})} D_{k_{4},m_{4}}^{(\ell_{5})} D_{k_{5},m_{5}}^{(\ell_{5})} \left(-1\right)^{m^{1}+m^{2}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ m_{1:2} & m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell^{2} \\ -m^{1} & m_{3} & m^{2} \end{pmatrix} \begin{pmatrix} \ell^{2} & \ell_{4:5} \\ -m^{2} & m_{4:5} \end{pmatrix} \\ \\ = \left(-1\right)^{k^{1}+k^{2}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ k_{1:2} & k^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell^{2} \\ -k^{1} & k_{3} & k^{2} \end{pmatrix} \begin{pmatrix} \ell^{2} & \ell_{4:5} \\ -k^{2} & k_{4:5} \end{pmatrix}.$$

Indeed, we have

$$D_{k_4,m_4}^{(\ell_4)}D_{k_5,m_5}^{(\ell_5)} = \sum_{\ell^2 \ k^2 \ m^2} \left(2\ell^2 + 1\right) \begin{pmatrix} \ell_{4:5} & \ell^2 \\ m_{4:5} & m^2 \end{pmatrix} \begin{pmatrix} \ell_{4:5} & \ell^2 \\ k_{4:5} & k^2 \end{pmatrix} D_{k^2,m^2}^{(\ell^2)*}$$

 $m_1 + m_2 + m_3 + m_4 = 0$ 

$$\mathcal{H}_{k_{1:5},m_{1:5}}^{\ell_{1:5}}\left(\ell^{1:2}\right) = \sum_{\ell^2,k^2,m^2} \sum_{m_{1:5}} D_{k_1,m_1}^{(\ell_1)} D_{k_2,m_2}^{(\ell_2)} D_{k_3,m_3}^{(\ell_3)} D_{k^2,m^2}^{\left(\ell^2\right)*} \left(-1\right)^{m^1+m^2} \left(2\ell^2+1\right)$$

$$\times \begin{pmatrix} \ell_{4:5} & \ell^2 \\ m_{4:5} & m^2 \end{pmatrix} \begin{pmatrix} \ell_{4:5} & \ell^2 \\ k_{4:5} & k^2 \end{pmatrix} \begin{pmatrix} \ell_{1:2} & \ell^1 \\ m_{1:2} & m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_3 & \ell^2 \\ -m^1 & m_3 & m^2 \end{pmatrix} \begin{pmatrix} \ell^2 & \ell_{4:5} \\ -m^2 & m_{4:5} \end{pmatrix},$$

now by (35)

$$\mathcal{H}_{k_{1:5},m_{1:5}}^{\ell_{1:5}}\left(\ell^{1:2}\right) = \sum_{\ell^{2},k^{2},m^{2}} \begin{pmatrix} \ell_{4:5} & \ell^{2} \\ k_{4:5} & k^{2} \end{pmatrix} (-1)^{k^{2}} \begin{pmatrix} 2\ell^{2}+1 \end{pmatrix} \sum_{m_{4:5}} \begin{pmatrix} \ell^{2} & \ell_{4:5} \\ -m^{2} & m_{4:5} \end{pmatrix} \begin{pmatrix} \ell_{4:5} & \ell^{2} \\ m_{4:5} & m^{2} \end{pmatrix} \\ \times \sum_{m_{1:3}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{k_{3},m_{3}}^{(\ell_{3})} D_{-k,-m}^{(\ell)} (-1)^{m^{1}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ m_{1:2} & m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell^{2} \\ -m^{1} & m_{3} & m^{2} \end{pmatrix}.$$

3j-symbols are orthogonal, see (30), using (38) we obtain

$$\mathcal{H}_{k_{1:5},m_{1:5}}^{\ell_{1:5}}\left(\ell^{1:2}\right) = \begin{pmatrix} \ell_{4:5} & \ell^{2} \\ k_{4:5} & k^{2} \end{pmatrix} (-1)^{k^{2}}$$

$$\times \sum_{m_{1:3}} D_{k_{1},m_{1}}^{(\ell_{1})} D_{k_{2},m_{2}}^{(\ell_{2})} D_{k_{3},m_{3}}^{(\ell_{3})} D_{-k^{2},-m^{2}}^{(\ell)} \left(-1\right)^{m^{1}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ m_{1:2} & m^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell^{2} \\ -m^{1} & m_{3} & -m^{2} \end{pmatrix}$$

$$= (-1)^{k^{1}+k^{2}} \begin{pmatrix} \ell_{1:2} & \ell^{1} \\ k_{1:2} & k^{1} \end{pmatrix} \begin{pmatrix} \ell^{1} & \ell_{3} & \ell^{2} \\ -k^{1} & k_{3} & k^{2} \end{pmatrix} \begin{pmatrix} \ell^{2} & \ell_{3:4} \\ -k^{2} & k_{3:4} \end{pmatrix},$$

observe that  $k^1$  and  $k^2$  are given by  $k_{1:2}$  and  $k_{3:4}$  respectively hence the signe of them can be chosen freely.

**Proof of Lemma.** 20 We have

$$\operatorname{Cum}_{p}\left(\mathcal{Z}_{\ell_{1:p}}^{m_{1:p}}\right) = \sum_{k_{1:p}} \prod_{a=1}^{p} D_{m_{a},k_{a}}^{(\ell_{a})} \operatorname{Cum}_{4}\left(Z_{\ell_{1:p}}^{k_{1:p}}\right).$$

Under isotropy assumption

$$\operatorname{Cum}_{p}\left(Z_{\ell_{1:p}}^{m_{1:p}}\right) = \sum_{k_{1:p}} \int_{SO(3)} \prod_{a=1}^{p} D_{m_{a},k_{a}}^{(\ell_{a})} dg \operatorname{Cum}_{p}\left(Z_{\ell_{1:p}}^{k_{1:p}}\right)$$

$$= \sum_{\ell^{1:p-3},k^{1:p-3}} (-1)^{\sum m^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$\times \sum_{k_{1:p}} (-1)^{\sum k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -k^{a} & k_{a+2} & k^{a+1} \end{pmatrix} \left(2\ell^{a+1} + 1\right) \operatorname{Cum}_{p}\left(Z_{\ell_{1:p}}^{k_{1:p}}\right)$$

$$= \sum_{\ell^{1:p-3},k^{1:p-3}} (-1)^{\sum m^{1:p-3}} \prod_{a=0}^{p-3} \sqrt{2\ell^{a} + 1} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix} \widetilde{S}_{p}\left(\ell_{1:p}|\ell^{1:p-3}\right).$$

If isotropy is not assumed then from

$$\operatorname{Cum}_{p}\left(Z_{\ell_{1:p}}^{k_{1:p}}\right) = \sum_{\ell^{1:p-3}, k^{1:p-3}} (-1)^{\sum k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -k^{a} & k_{a+2} & k^{a+1} \end{pmatrix}$$
$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^{a} + 1} \widetilde{S}_{p}\left(\ell_{1:p} | \ell^{1:p-3}\right),$$

we obtain

$$\operatorname{Cum}_{p}\left(\mathcal{Z}_{\ell_{1:p}}^{m_{1:p}}\right) = \sum_{k_{1:p}} \prod_{a=1}^{p} D_{m_{a},k_{a}}^{(\ell_{a})} \left(-1\right)^{\sum k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -k^{a} & k_{a+2} & k^{a+1} \end{pmatrix}$$

$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \widetilde{S}_{p} \left(\ell_{1:p} | \ell^{1:p-3}\right)$$

$$= (-1)^{\sum m^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^{a} & \ell_{a+2} & \ell^{a+1} \\ -m^{a} & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^{a}+1} \widetilde{S}_{p} \left(\ell_{1:p} | \ell^{1:p-3}\right),$$

see Lemma 31 .  $\blacksquare$ 

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