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ABSENCE OF SCALAR HAIR IN SCALAR-TENSOR GRAVITY

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Stationary, asymptotically flat black holes in scalar-tensor theories of gravity are studied. It is shown that such black holes have no scalar hair and are the same as in General Relativity.

Keywords: Black holes, scalar-tensor gravity

1. Introduction

In General Relativity (GR) stationary black holes, which are the endpoint of gravitational collapse, must be axisymmetric and are described by the Kerr-Newman metric.¹ The prototypical alternative theory of gravity is Brans-Dicke theory with (Jordan frame) action

$$S_{BD} = \int d^4x \sqrt{-\hat{g}} \left[\varphi \hat{R} - \frac{\omega_0}{\varphi} \hat{\nabla}^{\mu} \varphi \hat{\nabla}_{\mu} \varphi + L_m(\hat{g}_{\mu\nu}, \psi) \right] . \tag{1}$$

In 1972 Hawking showed that stationary black holes in this theory must be the Kerr-Newman black holes of GR.² This result was generalized by Bekenstein to more general scalar-tensor theories, but with the additional assumption of spherical symmetry.³ Hawking's result has recently been extended to general scalar-tensor theories with action

$$S_{ST} = \int d^4x \sqrt{-\hat{g}} \left[\varphi \hat{R} - \frac{\omega(\varphi)}{\varphi} \hat{\nabla}^{\mu} \varphi \hat{\nabla}_{\mu} \varphi - V(\varphi) + L_m(\hat{g}_{\mu\nu}, \psi) \right]$$
(2)

without any extra assumption of symmetry apart from stationarity.⁴ This proof is presented below.

2. The proof

To begin with, we require:

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- Asymptotic flatness: this requires $V(\varphi_0) = 0$ and $\varphi_0 V'(\varphi_0) = 2V(\varphi_0)$, where φ_0 is the value the Brans-Dicke scalar field approaches as $r \to +\infty$ (gravitational collapse occurs on scales much smaller than the Hubble scale H_0^{-1} , so asymptotic flatness is expected to be an adequate approximation physically).
- Stationarity: the black hole is supposed to be the endpoint of collapse.

We map the theory to the Einstein conformal frame according to $\hat{g}_{\mu\nu} \to g_{\mu\nu} = \varphi \, \hat{g}_{\mu\nu}, \, \varphi \to \phi$, with $d\phi = \sqrt{\frac{2\omega(\varphi)+3}{16\pi}} \, \frac{d\varphi}{\varphi}$ (for $\omega \neq -3/2$). The action becomes

$$S_{ST} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi} - \frac{1}{2} \nabla^{\mu} \phi \nabla_{\mu} \phi - U(\phi) + L_m(\hat{g}_{\mu\nu}, \psi) \right], \tag{3}$$

where $U(\phi) = V(\varphi)/\varphi^2$. The field equation for the scalar in vacuo in the Einstein frame is

$$\Box \phi = U'(\phi) \,. \tag{4}$$

Since the conformal factor of the transformation depends only on the Brans-Dicke field φ , the Einstein frame symmetries are the same as in the Jordan frame. In particular, there exists a Killing vector ξ^{μ} which is timelike at infinity (stationarity). In the Einstein frame and in electrovacuum the theory is essentially GR with a minimally coupled scalar field. So, stationary, asymptotically flat black holes have to be axisymmetric and, hence, there should be a second Killing vector ζ^{μ} which is spacelike at infinity, provided that the stress-energy tensor for φ satisfies the weak energy condition. Consider, in vacuo, a 4-volume \mathcal{V} bounded by the horizon H, two partial Cauchy hypersurfaces \mathcal{S}_1 , \mathcal{S}_2 , and a timelike 3-surface at infinity. Now multiply both sides of eq. (4) by U' and integrate over the 4-volume \mathcal{V} , obtaining

$$\int_{\mathcal{V}} d^4x \sqrt{-g} \, U'(\phi) \Box \phi = \int_{\mathcal{V}} d^4x \sqrt{-g} \, U'^2(\phi) \,. \tag{5}$$

We can rewrite this equation as

$$\int_{\mathcal{V}} d^4x \sqrt{-g} \left[U''(\phi) \nabla^{\mu} \phi \nabla_{\mu} \phi + U'^2(\phi) \right] = \int_{\partial \mathcal{V}} d^3x \sqrt{|h|} U'(\phi) n^{\mu} \nabla_{\mu} \phi , \qquad (6)$$

where n^{μ} is the normal to the boundary and h is the determinant of the induced metric $h_{\mu\nu}$ on this boundary. Splitting the boundary into its constituent parts one has $\int_{r=\infty} = 0$,

$$\int_{horizon} d^3x \sqrt{|h|} \, U'(\phi) n^{\mu} \nabla_{\mu} \phi = 0 \,, \tag{7}$$

because of the spacetime symmetries, and $\int_{\mathcal{S}_1} = -\int_{\mathcal{S}_2}$ if S_2 is obtained by shifting each point of S_1 along integral curves of ξ^{μ} , hence

$$\int_{\mathcal{V}} d^4x \sqrt{-g} \left[U''(\phi) \nabla^{\mu} \phi \nabla_{\mu} \phi + U'^2(\phi) \right] = 0.$$
 (8)

 $U'^2 \geq 0$, $\nabla^{\mu}\phi$ (which is orthogonal to both ξ^{μ} and ζ^{μ}) is spacelike or zero, and with $U''(\phi) \geq 0$ being a local stability condition, one concludes that it must be $\nabla_{\mu}\phi \equiv 0$

in \mathcal{V} and $U'(\phi_0) = 0$. But for $\phi = \text{const.}$, to which we have reduced, the scalar-tensor theory reduces to GR and the black hole must be described by the Kerr metric.

Metric f(R) gravity, which has seen much recent attention,^{5,6} is a special Brans-Dicke theory with parameter $\omega = 0$ and a non-trivial potential V for the Brans-Dicke field $\varphi = f'(R)$. Palatini f(R) gravity, instead, corresponds to an $\omega = -3/2$ Brans-Dicke theory (again, with a potential). The case $\omega = -3/2$ was explicitly excluded in our discussion but $\omega = -3/2$ Brans-Dicke theory reduces to GR in vacuo anyway.

3. Conclusions

The proof presented above extends immediately to electro-vacuum and to any form of conformal matter with trace of the energy-momentum tensor T=0. It implies that asymptotically flat black holes that are the endpoint of collapse in scalar-tensor gravity are described by the Kerr-Newman metric. The assumption of asymptotic flatness is a limitation mathematically, but one expects on physical grounds that the effect of a Friedmann-Lemaître-Robertson-Walker asymptotic structure on astrophysical collapse to be completely negligible (except for primordial black holes for which the collapse and the Hubble scales can be comparable⁷).

There are certain exceptions to the proof, which include: (i) theories in which $\omega \to \infty$ somewhere outside or on the horizon; (ii) theories in which $\varphi \to \infty$ or $\varphi \to 0$ somewhere outside or on the horizon; ^a (iii) theories in which the stressenergy tensor of the Einstein-frame scalar violates the weak energy condition.

It is likely that the majority of these exceptional theories or solutions will be unphysical (e.g., the gravitational coupling in scalar-tensor gravity is inversely proportional to φ) but interesting exceptions might exist. This issue will be addressed in future work.

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References

- 1. S.W. Hawking, Commun. Math. Phys. 25, 152 (1972).
- 2. S.W. Hawking, Comm. Math. Phys. 25, 167 (1972).
- 3. A.E. Mayo and J.D. Bekenstein, *Phys. Rev. D* **54**, 5059 (1996); J.D. Bekenstein, arXiv:gr-qc/9605059.
- 4. T.P. Sotiriou and V. Faraoni, Phys. Rev. Lett. 108, 081103 (2012).

^aAn example of a solution where $\varphi \to \infty$ on the horizon is that of Bocharova *et al.*⁸ (which is, however, unstable⁹).

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- 5. T.P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010).
- 6. A. De Felice and S. Tsujikawa, Living Rev. Rel. 13, 3 (2010).
- 7. T. Jacobson, Phys. Rev. Lett. 83, 2699 (1999).
- 8. N. Bocharova, K. Bronnikov, and V. Melnikov, Vestn. Mosk. Univ. Fiz. Astron. 6, 706 (1970).
- 9. K.A. Bronnikov and Yu.N. Kireyev, Phys. Lett. A $\mathbf{67}$, 95 (1978).