Bimetric Theory and Partial Masslessness with Lanczos-Lovelock Terms in Arbitrary Dimensions

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ABSTRACT: Ghost-free bimetric theories describe nonlinear interactions of massive and massless spin-2 fields and, hence, provide a natural framework for investigating the phenomenon of partial masslessness for massive spin-2 fields at the nonlinear level. In this paper we analyze the spectrum of the ghost-free bimetric theory in arbitrary dimensions. Using a recently proposed construction, we identify the candidate nonlinear partially massless (PM) theories. It is shown that, in a 2-derivative setup, nonlinear PM theories can exist only in 3 and 4 dimensions. But on adding Lanczos-Lovelock terms to the bimetric action it is found that higher derivative nonlinear PM theories also exist in higher dimensions. This is consistent with existing results on the direct construction of cubic vertices with PM gauge symmetry. We obtain the candidate nonlinear PM theories in 5, 6 and 8 dimensions but show that none exist in 7 dimensions.

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1 Introduction

To focus attention and fix conventions, we start with a review of partial masslessness in linear Fierz-Pauli (FP) theory and the question of its nonlinear extension. We then describe our approach to the problem and summarize our results.

1.1 Partial masslessness in Fierz-Pauli theory and beyond

The linear fluctuations $h_{\mu\nu}$ of a massive spin-2 field are governed by the Fierz-Pauli equation [1, 2] which in a d-dimensional spacetime with a background metric $\bar{G}_{\mu\nu}$ becomes,

$$\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} - \frac{2}{d-2} \tilde{\Lambda} \left(h_{\mu\nu} - \frac{1}{2} \bar{G}_{\mu\nu} \bar{G}^{\rho\sigma} h_{\rho\sigma} \right) + \frac{\tilde{m}_{\rm FP}^2}{2} \left(h_{\mu\nu} - \bar{G}_{\mu\nu} \bar{G}^{\rho\sigma} h_{\rho\sigma} \right) = 0. \tag{1.1}$$

 $\tilde{\Lambda}$ is the cosmological constant and \tilde{m}_{FP} is the mass. For $\tilde{m}_{FP} = 0$, (1.1) must reduce to the linearized Einstein equation for a massless spin-2 field, hence $\tilde{\mathcal{E}}$ is given by

$$\tilde{\mathcal{E}}_{\mu\nu}^{\rho\sigma}h_{\rho\sigma} = -\frac{1}{2} \left[\delta_{\mu}^{\rho}\delta_{\nu}^{\sigma}\nabla^{2} + \bar{G}^{\rho\sigma}\nabla_{\mu}\nabla_{\nu} - \delta_{\mu}^{\rho}\nabla^{\sigma}\nabla_{\nu} - \delta_{\nu}^{\rho}\nabla^{\sigma}\nabla_{\mu} - \bar{G}_{\mu\nu}\bar{G}^{\rho\sigma}\nabla^{2} + \bar{G}_{\mu\nu}\nabla^{\rho}\nabla^{\sigma} \right] h_{\rho\sigma} . \quad (1.2)$$

The structure of (1.1) is determined by the requirement of the absence of ghost. Generically, it allows for d(d-1)/2-1 propagating modes corresponding to a massive spin-2 field in d dimensions. The nature and dynamics of $\bar{G}_{\mu\nu}$ cannot be specified further in this framework, but a nonlinear setup that embeds (1.1) must address this issue.

If $\bar{G}_{\mu\nu}$ is a dS or, in general, an Einstein spacetime, then on the Higuchi bound [3, 4],

$$\tilde{m}_{\rm FP}^2 = \frac{2}{d-1}\tilde{\Lambda}\,,\tag{1.3}$$

equation (1.1) develops a gauge invariance $h_{\mu\nu} \longrightarrow h_{\mu\nu} + \Delta h_{\mu\nu}$ with [5–10],

$$\Delta h_{\mu\nu} \equiv \left(\nabla_{\mu}\nabla_{\nu} + \frac{2\Lambda}{(d-1)(d-2)}\bar{G}_{\mu\nu}\right)\xi(x). \tag{1.4}$$

This can be used to gauge away the helicity zero component of $h_{\mu\nu}$, so in d=4 only the four polarizations $\pm 2, \pm 1$ survive. This is the linear "partially massless" (PM) theory.¹

The obvious question of course is if the gauge symmetry associated with partial masslessness can be generalized from the FP theory in dS spacetimes to a nonlinear theory of spin-2 fields. Several recent studies have attempted to address this issue using different approaches [12, 14–16]. In particular, the authors in [12, 14] directly construct cubic vertices for $h_{\mu\nu}$ with the above PM gauge invariance. This constructive approach makes interesting predictions about the nonlinear PM theory, showing that:

- Cubic $h_{\mu\nu}$ interactions with a PM gauge invariance (1.4) exist only in 3 [12] and 4 [12, 14] dimensions, as long as the theory involves no more than 2 derivatives. Hence, while linear PM theory exists in any dimension, 2-derivative nonlinear PM theories can exist only in 3 and 4 dimensions.
- When higher derivative interactions are allowed, then PM gauge invariant cubic terms can be constructed even for dimensions d > 4 [12]. The structure of the higher derivative terms is such that for d = 4 one again recovers the 2-derivative theory. Hence the higher derivative terms are relevant only for d > 4.

Any method of constructing nonlinear PM theories must give rise to the above features implied by the explicit cubic vertex calculations.

In [17] we obtained a potential nonlinear PM theory as a special ghost-free bimetric theory [18] in d=4. There it was not obvious that this construction was consistent with the above dimension dependent features of PM theories. In the present paper, we implement this construction in arbitrary dimensions obtaining candidate PM theories including higher derivative terms. It is found that the construction indeed meets the expectations from the cubic vertex analysis. Let us consider this in some detail.

It is natural to look for nonlinear PM theories among the nonlinear extensions of the FP theory (1.1) that do not suffer from the Boulware-Deser ghost instability [19, 20]. For

¹PM theories also arise in the more general context of higher-spin theories, see for example, [11–13]. But here we concentrate only on the spin-2 case.

 $\bar{G}_{\mu\nu} = \eta_{\mu\nu}$ such a nonlinear theory was obtained and shown to be ghost-free in a certain "decoupling limit" in [21, 22] (also see [23]), whereas the complete nonlinear proof of absence of ghost was given in [24, 25]. However, these models do not admit dS solutions with massive FP perturbations around them. A generalization with extra parameters that accommodates a non-flat but non-dynamical $\bar{G}_{\mu\nu}$ was first considered in [26]. In this case decoupling limit arguments do not exist to argue the absence of the BD ghost, but unitarity was proven directly in the nonlinear theory in a Hamiltonian analysis [25, 27, 28]. Finally, the completely dynamical ghost-free theory was given in [18], as a bimetric theory conveniently formulated in terms of two interacting spin-2 fields $g_{\mu\nu}$ and $f_{\mu\nu}$. This theory contains the two constraints required to eliminate the BD ghost [18, 25]. For related work see [29–48]

As such, neither of the two metrics in the bimetric theory corresponds directly to the massive fluctuation in (1.1). But the theory can be reformulated in terms of a massive spin-2 field $M_{\mu\nu}^G$ coupled to a massless metric $G_{\mu\nu}$ such that linear perturbations $\delta M_{\mu\nu}^G$ around a background $\bar{M}_{\mu\nu}^G = 0$ always satisfy the FP equation (1.1) where now one identifies $h_{\mu\nu} = \delta M_{\mu\nu}^G$ [49]. Furthermore, (1.1) is also supplemented by a massless equation for $\delta G_{\mu\nu}$, while $\bar{G}_{\mu\nu}$ is determined as a solution to an ordinary Einstein's equation. In the absence of matter couplings, a background $\bar{M}_{\mu\nu}^G = 0$ forces $\bar{G}_{\mu\nu}$ to be an Einstein metric.² Hence, the bimetric theory provides a completely dynamical and ghost-free nonlinear extension of (1.1). For this reason it also provides the natural arena for investigating partial masslessness beyond the FP theory.

Bimetric theories contain several parameters that can be easily tuned to put equation (1.1) in the dS backgrounds on the Higuchi bound (1.3). The linear theory will now have the gauge symmetry (1.4), but only the $\xi = constant$ part of this transformation preserves the dS backgrounds. In [17] it was argued that the consistency of these dS preserving transformations with the dynamical nature of the backgrounds leads to a criterion for partial masslessness that is powerful enough to fix most of the remaining parameters of the model. The criterion was implemented in d=4 leading to a unique class of nonlinear candidate PM theories [17]. The complete gauge symmetry of the nonlinear model is not yet known but some special cases can be considered. Here we implement this criterion in arbitrary dimensions to obtain candidate PM theories that are consistent with expectations from the cubic vertex calculations described above.

1.2 Summary of results

Before getting into technical details, let us briefly summarize our results.

• The bimetric theory has a well defined mass spectrum around proportional backgrounds $f_{\mu\nu} = c^2 g_{\mu\nu}$ where c is generically determined by the equations of motion. We obtain the mass spectrum of the fluctuations in arbitrary dimensions with explicit expressions for the mass and the cosmological constant. We also construct the nonlinear extensions

²The theory has a much more complicated behaviour around backgrounds with $\bar{M}_{\mu\nu}^G \neq 0$ but this is not relevant for comparison to the FP equation.

of the massless and massive fluctuations which are useful for the purpose of fixing the normalizations in the PM case.

- We then discuss the linear PM theory in the context of bimetric theory linearized around dS backgrounds. The consistency of a subset of (1.4) with the dynamical backgrounds leads to a simple criterion for identifying nonlinear PM theories as special bimetric theories with the parameters fixed such that the equations of motion leave the c in the ansatz $f_{\mu\nu} = c^2 g_{\mu\nu}$ undetermined. In the PM theory c can be traded off with a constant gauge parameter. Applying this criterion to the 2-derivative bimetric action we find that nonlinear PM theories can exist only in d=3 and d=4. Although the full gauge transformation of nonlinear PM theories is not yet known, we show that the mass, cosmological constant, couplings and the background metric $\bar{G}_{\mu\nu}$ all become c-independent, and hence gauge invariant, for the PM parameter values.
- To explore the possibility of recovering nonlinear PM theories for d > 4 with the help of higher derivative terms, we consider the bimetric theory with extra Lanczos-Lovelock terms for both metrics.³ The analysis of the spectrum around dS backgrounds can easily generalized to the LL terms using recent results of [51] obtained for the LL extension of the Einstein-Hilbert gravity. We show that the construction outlined above can now be carried out leading to potential nonlinear PM theories even for d > 4. In particular, we obtain such nonlinear theory for d = 5, 6, 8 but show that no PM theory exists for d = 7.
- We also outline the structure of the bimetric action expressed in terms of the nonlinear extensions of the massless and the massive fields. This will be useful for comparison with the outcome of the direct cubic vertex construction, although such an explicit comparison has not been attempted here.

2 Bimetric description of massless and massive spin-2 fields in d dimensions

In this section we consider the ghost-free bimetric theory in d dimensions and describe in what sense it is a theory of a massive and a massless spin-2 fields. To this end, we consider a class of bimetric backgrounds, the proportional backgrounds, around which linear perturbations decompose into well defined massless and massive modes. These can then be promoted to nonlinear fields, generalizing the notion of massless and massive spin-2 fields beyond the special class of proportional backgrounds. The analysis generalizes the d=4 case considered in [49].

³Such an extension has been considered earlier in a different context in [50].

2.1 The bimetric theory in d dimensions

The ghost-free bimetric action for two spin-2 field $g_{\mu\nu}$ and $f_{\mu\nu}$ [18] can be easily generalized to arbitrary dimensions,⁴

$$S_{gf} = \int d^d x \left[m_g^{d-2} \sqrt{|g|} R(g) + m_f^{d-2} \sqrt{|f|} R(f) - 2m^d \sqrt{|g|} V(S; \beta_n) \right], \qquad (2.1)$$

where $\sqrt{|g|} = \sqrt{|\det g|}$ and S simply stands for the matrix square-root,

$$S \equiv \sqrt{g^{-1}f} \,. \tag{2.2}$$

The interaction potential is given by,

$$V(S; \beta_n) = \sum_{n=0}^{d} \beta_n e_n(S), \qquad (2.3)$$

where $e_n(S)$ are the elementary symmetric polynomials of eigenvalues of S. They are expressible as polynomials of $\text{Tr}(S^k)$ and can be iteratively constructed starting with $e_0(S) = 1$ and using Newton's identities,

$$e_n(S) = -\frac{1}{n} \sum_{k=1}^n (-1)^k \operatorname{Tr}(S^k) e_{n-k}(S).$$
 (2.4)

In particular, $e_d(S) = \det S$ and $e_n(S) = 0$ for n > d.

The β_n in (2.3) and the Planck masses m_g and m_f are the d+3 free parameters of the theory. The mass parameter m is degenerate with the overall scale of the β_n . The action contains the cosmological terms $\beta_0 \sqrt{|g|}$ and $\beta_d \sqrt{|f|}$, but the actual cosmological constants are to be read off from the equations of motion.

The absence of the Boulware-Deser ghost in bimetric theory in d=4 was proved in [18, 25]. It is straightforward to extend the ghost analysis to arbitrary dimensions, especially using the "deformed determinant" representation for the potential [26]. Setting $f_{\mu\nu}$ to a non-dynamical flat metric and tuning the β_n to exclude a cosmological constant contribution in a flat $g_{\mu\nu}$ background, one recovers the massive gravity model of [22].

The bimetric potential (2.3) has the following useful symmetry property under the interchange of $g_{\mu\nu}$ and $f_{\mu\nu}$ that sends $S \to S^{-1}$ [18],

$$\sqrt{|g|} V(S; \beta_n) = \sqrt{|f|} V(S^{-1}; \beta_{d-n}).$$
(2.5)

This allows us to obtain the f-sector equations from the g-sector of the theory.

The sourceless equations of motion obtained on varying the action (2.1) with respect to $g^{\mu\nu}$ and $f^{\mu\nu}$ are,

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) + \frac{m^d}{m_a^{d-2}}V_{\mu\nu}^g = 0, \qquad R_{\mu\nu}(f) - \frac{1}{2}f_{\mu\nu}R(f) + \frac{m^d}{m_e^{d-2}}V_{\mu\nu}^f = 0.$$
 (2.6)

⁴In the paper we use the sign and curvature conventions of Wald [52] so that $R_{\mu\nu} = \partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} + \cdots$

The potential contributions have the form,

$$V_{\mu\nu}^g = g_{\mu\lambda} V_{\nu}^{\lambda g}(S), \qquad V_{\mu\nu}^f = f_{\mu\lambda} V_{\nu}^{\lambda f}(S^{-1}),$$
 (2.7)

and are explicitly given by (A.8) and (A.9) in the the appendix (where the details of the derivation are also provided).

2.2 The proportional backgrounds

Let us now consider a class of solutions to the bimetric equations in which the metrics are proportional to each other,

$$\bar{f}_{\mu\nu} = c^2 \, \bar{g}_{\mu\nu} \,.$$
 (2.8)

Such backgrounds have two important properties: 1) they are the most general class of backgrounds around which the bimetric theory has well defined massless and massive fluctuations with a Fierz-Pauli mass term, 2) they coincide with classical solutions in general relativity.⁵ Indeed, for this ansatz, the bimetric equations (2.6) imply that c is constant and then reduce to two copies of cosmological Einstein's equation for $\bar{g}_{\mu\nu}$,

$$R_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}R(\bar{g}) + \bar{g}_{\mu\nu}\Lambda_g = 0, \qquad R_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}R(\bar{g}) + \bar{g}_{\mu\nu}\Lambda_f = 0.$$
 (2.9)

The cosmological constants are given by (for details see the appendix),

$$\Lambda_g = \frac{m^d}{m_g^{d-2}} \sum_{n=0}^{d-1} {d-1 \choose n} c^n \beta_n , \qquad \Lambda_f = \frac{m^d}{m_f^{d-2}} c^{2-d} \sum_{n=1}^d {d-1 \choose n-1} c^n \beta_n , \qquad (2.10)$$

where, $\binom{n}{k} = \frac{n!}{k!(n-k!)}$ is the combinatorial factor. Note that Λ_g does not contain β_d while Λ_f is independent of β_0 .

The consistency of the two equations in (2.9) with each other then implies,⁶

$$\Lambda_q = \Lambda_f \,. \tag{2.11}$$

This is a polynomial equation that, generically, determines c in terms of the parameters of the theory, the exception being the PM case [17] to be discussed later. In particular, the background equations (2.9) admit de Sitter solutions that are relevant for the identification of the linear PM theory.

Since (2.11) is homogeneous in the β_k , the c determined by it is independent of the overall scale of the β_k and will depend on at most d+1 parameters, say $\beta_1/\beta_0, \dots \beta_d/\beta_0$ and $\alpha = m_f/m_g$. We are interested in parameter values that result in real non-zero c, as c also appears in other quantities, like the Fierz-Pauli mass of the fluctuations. In practice, such parameter ranges can be easily found since, given any d of the parameters, say, $\{\alpha, \beta_2/\beta_0, \dots, \beta_d/\beta_0\}$, one can determine the allowed range of the remaining one, β_1/β_0 , by expressing it as a function of the real c using (2.11).

⁵For another feature of such solutions, see [53]

⁶The discussion can be easily extended to include sources, in which case, (2.8) also forces a proportionality relation between the two energy-momentum tensors as discussed in [49].

2.3 Linear mass eigenstates

To determine the mass spectrum of the theory, let us now consider linear perturbations around the proportional backgrounds,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,, \quad f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu} \,,$$
 (2.12)

with $\bar{f}_{\mu\nu} = c^2 \bar{g}_{\mu\nu}$. It turns out that only around such backgrounds the fluctuations can be combined into definite mass eigenstates. A massive mode in a general background is identified through the appearance of a Fierz-Pauli mass term. On linearizing the equations of motion (2.6), one gets (for details, see the appendix)

$$\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta g_{\rho\sigma} - \frac{2}{d-2}\Lambda_g \left(\delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta g_{\rho\sigma}\right) + \frac{m^d}{m_g^{d-2}}N\,\bar{g}_{\mu\lambda}\,\left(\mathrm{Tr}(\delta S)\delta^{\lambda}_{\nu} - \delta S^{\lambda}_{\nu}\right) = 0\,,\tag{2.13}$$

$$\bar{\mathcal{E}}_{\mu\nu}^{\rho\sigma}\delta f_{\rho\sigma} - \frac{2}{d-2}\Lambda_g \left(\delta f_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta f_{\rho\sigma}\right) - \frac{m^d}{m_f^{d-2}}Nc^{4-d}\bar{g}_{\mu\lambda} \left(\text{Tr}(\delta S)\delta_{\nu}^{\lambda} - \delta S_{\nu}^{\lambda}\right) = 0. \quad (2.14)$$

 $\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}$ is defined through (1.2), now with background metric $\bar{g}_{\mu\nu}$, and N is given by (A.26). $\delta S^{\mu}_{\ \nu} = \frac{1}{2c} \bar{g}^{\mu\lambda} (\delta f - c^2 \delta g)_{\lambda\nu}$ enters both equations in the FP combination, hence we expect to get a massive mode $\delta M_{\mu\nu} \sim \bar{g}_{\mu\lambda} \delta S^{\lambda}_{\ \nu}$.

The linearized equations are easily diagonalized in terms of a massless fluctuation $\delta G_{\mu\nu}$ and a massive fluctuation $\delta M_{\mu\nu}$,

$$\delta G_{\mu\nu} = A(c) \left(\delta g_{\mu\nu} + c^{d-4} \alpha^{d-2} \delta f_{\mu\nu} \right) , \qquad (2.15)$$

$$\delta M_{\mu\nu} = \frac{B(c)}{2c} \left(\delta f_{\mu\nu} - c^2 \delta g_{\mu\nu} \right) . \tag{2.16}$$

A(c) and B(c) are normalizations to be determined later and we have used the notation,

$$\alpha = \frac{m_f}{m_g}. (2.17)$$

Indeed, adding (2.13) and (2.14) to cancel the FP mass term gives a massless spin-2 equation,

$$\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta G_{\rho\sigma} - \frac{2}{d-2}\Lambda_g \left(\delta G_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta G_{\rho\sigma}\right) = 0.$$
 (2.18)

On the other hand, subtracting the right combination of (2.13) and (2.14) gives the FP equation (1.1) for the massive spin-2 fluctuation,

$$\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\,\delta M_{\rho\sigma} - \frac{2}{d-2}\Lambda_g\left(\delta M_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta M_{\rho\sigma}\right) + \frac{m_{\rm FP}^2}{2}\left(\delta M_{\mu\nu} - \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta M_{\rho\sigma}\right) = 0. \tag{2.19}$$

Here, the FP mass is given by,

$$m_{\rm FP}^2 = \frac{m^d}{m_g^{d-2}} \left(\frac{1 + (\alpha c)^{d-2}}{(\alpha c)^{d-2}} \right) \sum_{n=1}^{d-1} {d-2 \choose n-1} c^n \beta_n . \tag{2.20}$$

We emphasize that equations (2.18) (2.19) are written with $\bar{g}_{\mu\nu}$ as the background metric. Then the expressions for $m_{\rm FP}^2$ (2.20) and Λ_g (2.10) refer to this background metric choice.

2.4 The nonlinear massive and massless fields

The massless and massive fluctuations can be regarded as perturbations of some nonlinear fields $G_{\mu\nu}$ and $M_{\mu\nu}^G$. If this is the only criterion, the choice of nonlinear fields is far from unique. However, we also require that the relation between the nonlinear fields and the original variables $g_{\mu\nu}$ and $f_{\mu\nu}$ is simple enough that it is invertible in a useful way [49]. These criteria single out the following straightforward nonlinear extensions of the mass eigenstates,

$$G_{\mu\nu} = A(c) \left(g_{\mu\nu} + c^{d-4} \alpha^{d-2} f_{\mu\nu} \right),$$
 (2.21)

$$M_{\mu\nu}^{G} = B(c) \left(G_{\mu\rho} S_{\nu}^{\rho} - c G_{\mu\nu} \right) . \tag{2.22}$$

where $S = \sqrt{g^{-1}f}$ and the dimension dependent normalizations A and B will be fixed later in the context of the PM theory. It is not claimed that G and M^G propagate respectively 2 and 5 degrees of freedom nonlinearly.

The bimetric action (2.1) can be re-expressed in terms of the new nonlinear fields as a theory of a massive spin-2 field $M_{\mu\nu}^G$ interacting with a massless spin-2 field $G_{\mu\nu}$. However, it turns out that $G_{\mu\nu}$ cannot be coupled to matter in the standard way without reintroducing the Boulware-Deser ghost and, hence, it cannot be regarded as the physical gravitational metric [49]. In other words, the spin-2 mass eigenstates differ from the states produced by bimetric interactions, as is familiar from other contexts in particle physics.⁷ However, the theory in terms of G and M^G is relevant for discussing partial masslessness which is known to arise only in the absence of matter couplings.

On the proportional backgrounds, the nonlinear fields reduce to,

$$\bar{G}_{\mu\nu} = A(c) \left(1 + (\alpha c)^{d-2} \right) \bar{g}_{\mu\nu} , \qquad \bar{M}_{\mu\nu}^G = 0 .$$
 (2.23)

A non-vanishing expectation value for $M_{\mu\nu}^G$ signals non-proportional backgrounds and hence parametrizes deviations of the bimetric solutions from solutions in general relativity.

The linearized equations (2.18), (2.19) are written with $\bar{g}_{\mu\nu}$ as the background metric. To describe the theory entirely in terms of the massless and massive modes, we rewrite these in terms of $\bar{G}_{\mu\nu}$ using (2.23),

$$\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\,\delta G_{\rho\sigma} - \frac{2}{d-2}\tilde{\Lambda}_g\left(\delta G_{\mu\nu} - \frac{1}{2}\bar{G}_{\mu\nu}\bar{G}^{\rho\sigma}\delta G_{\rho\sigma}\right) = 0\,,\tag{2.24}$$

$$\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\,\delta M_{\rho\sigma} - \frac{2}{d-2}\tilde{\Lambda}_g\left(\delta M_{\mu\nu} - \frac{1}{2}\bar{G}_{\mu\nu}\bar{G}^{\rho\sigma}\delta M_{\rho\sigma}\right) + \frac{\tilde{m}_{\rm FP}^2}{2}\left(\delta M_{\mu\nu} - \bar{G}_{\mu\nu}\bar{G}^{\rho\sigma}\delta M_{\rho\sigma}\right) = 0. \quad (2.25)$$

Here $\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu}$ is given by (1.2). In the new background convention, the mass and the cosmological constant are rescaled to,

$$\tilde{m}_{\text{FP}}^2 = \frac{m_{\text{FP}}^2}{A(c)(1 + (\alpha c)^{d-2})}, \qquad \tilde{\Lambda}_g = \frac{\Lambda_g}{A(c)(1 + (\alpha c)^{d-2})}.$$
 (2.26)

⁷To describe a massive spin-2 field interacting with a gravitational metric that is also sourced by conventional matter, one should use the fields $g_{\mu\nu}$ and $M_{\mu\nu} = g_{\mu\lambda}S^{\lambda}_{\nu} - cg_{\mu\nu}$ [49].

3 Partial masslessness and Bimetric theory in arbitrary dimensions

To summarize, on linearizing, the bimetric action leads to the FP equation for massive spin-2 fields in a cosmological background (2.25), along with equation (2.24) for massless fluctuations of the background itself. Hence it provides a completely dynamical nonlinear generalization of the FP equation discussed in section 1.1. In particular, in this setup, one can investigate partial masslessness beyond the linear level.

Working in 4-dimensions, ref. [17] obtained a criterion to identify the nonlinear PM theory among the set of ghost-free bimetric theories. Here the considerations are extended to d-dimensions showing that, while linear PM theories exist for any d, nonlinear PM theories based on the bimetric action (2.1) exist only in d=3 and d=4. This is consistent with the results of [12] which shows that, in a 2-derivative theory, spin-2 cubic interactions invariant under the linear PM gauge symmetry exist only in these dimensions. Furthermore, [12] shows that when the 2-derivative restriction is relaxed, cubic terms with PM symmetry can be constructed in higher dimensions as well. This case is considered in the next section.

It should be emphasized that at this stage we do not identify the PM gauge symmetry in the nonlinear theory, except for a subset of the transformations around maximally symmetric backgrounds. Hence the claim is that if a nonlinear theory with a PM gauge symmetry exits, it must belong to the set of theories identified here.

3.1 Linear PM symmetry in bimetric variables

Since equation (2.25) coincides with the FP equation (1.1), it is straightforward to identify the linear PM theory and the associated gauge symmetry (1.4) in bimetric theory. Hence, when the Higuchi bound is satisfied,

$$\tilde{m}_{\rm FP}^2 = \frac{2}{d-1}\,\tilde{\Lambda}_g\,,\tag{3.1}$$

equation (2.25) develops a new gauge invariance that transforms the massive spin-2 fluctuation as $\delta M_{\mu\nu} \to \delta M_{\mu\nu} + \Delta(\delta M_{\mu\nu})$, where

$$\Delta(\delta M_{\mu\nu}) = \left(\nabla_{\mu}\nabla_{\nu} + \frac{2\tilde{\Lambda}_g}{(d-1)(d-2)}\bar{G}_{\mu\nu}\right)\xi(x). \tag{3.2}$$

Now we also have an equation (2.24) for the massless fluctuations $\delta G_{\mu\nu}$. Since the massless equation has no unusual symmetries, the above transformation must be supplemented by,

$$\Delta(\delta G_{\mu\nu}) = 0 \tag{3.3}$$

Using (2.15) and (2.16), one obtains the corresponding transformations of δg and δf as,

$$\Delta(\delta g_{\mu\nu}) = -\frac{2}{cB} \frac{(\alpha c)^{d-2}}{1 + (\alpha c)^{d-2}} \Delta(\delta M_{\mu\nu}), \qquad \Delta(\delta f_{\mu\nu}) = \frac{2c}{B} \frac{1}{1 + (\alpha c)^{d-2}} \Delta(\delta M_{\mu\nu}).$$
 (3.4)

In any dimension, these equations realize the linear PM gauge transformations in the bimetric theory around proportional backgrounds. The associated Higuchi bound (3.1) simply reduces the number of bimetric parameters by just one and it too can be satisfied in any dimension. But we will see that the nonlinear extension is much more restrictive.

3.2 Identifying the nonlinear PM theory in d dimensions

The gauge transformations (3.4) of the linear PM theory lead to the criteria for identifying potential nonlinear PM theories provided one works in a setup that treats the backgrounds dynamically. The argument is as follows [17]: A nonlinear PM theory must be invariant under an extension of (3.4) that transforms the nonlinear fields as, say,

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \Delta g_{\mu\nu} \,, \qquad f_{\mu\nu} \to f'_{\mu\nu} = f_{\mu\nu} + \Delta f_{\mu\nu} \,.$$
 (3.5)

For $g = \bar{g} + \delta g$, the transformed field also splits as $g' = \bar{g}' + \delta g'$, but there is a gauge ambiguity depending on how Δg is split. For instance, a small Δg may be viewed as transforming either the background, $\bar{g}' = \bar{g} + \Delta g$, or the fluctuation, $\delta g' = \delta g + \Delta g$. The same holds for f.

The implication of this for the linearized bimetric theory is that the variation $\Delta(\delta g)$ of δg (3.4) can, alternatively, be regarded as a transformation of the background \bar{g} to $\bar{g}' = \bar{g} + \Delta(\delta g)$, keeping δg unchanged. The same holds for $f_{\mu\nu}$. Let's adopt this latter point of view. Now, for a generic gauge parameter $\xi(x)$ in (3.2), the new (\bar{g}', \bar{f}') are not proportional backgrounds and hence are not dS metrics, in which case not much is known about partial masslessness. To keep to dS backgrounds we restrict $\Delta(\delta g)$ and $\Delta(\delta f)$ to constant $\xi(x) = \xi_0$ and rename them to $\Delta_0(\bar{g})$, $\Delta_0(\bar{f})$ to emphasize that they transform (\bar{g}, \bar{f}) rather than $(\delta g, \delta f)$. Then, given $\bar{f} = c^2 \bar{g}$, the new backgrounds are related through,

$$\bar{f}' = c'^2(\xi_0) \, \bar{g}' \,, \tag{3.6}$$

and the dS preserving gauge transformations at background level are,

$$\bar{g}' = \bar{g} + \Delta_0(\bar{g}), \qquad \bar{f}' = \bar{f} + \Delta_0(\bar{f}), \qquad c' = c + \Delta_0(c).$$
 (3.7)

The Δ_0 -variations above are obtained from (3.4), (3.2) and (3.6) for constant $\xi(x) = \xi_0$,

$$\Delta_0(\bar{g}) = -\frac{2}{c} (\alpha c)^{d-2} \, \bar{g} \, \lambda_d \, \xi_0 \,, \quad \Delta_0(\bar{f}) = 2c \, \bar{g} \, \lambda_d \, \xi_0 \,, \quad \Delta_0(c) = \left[1 + (\alpha c)^{d-2} \right] \lambda_d \, \xi_0 \,, \quad (3.8)$$

where, λ_d stands for,

$$\lambda_d = \frac{A}{B} \frac{2\tilde{\Lambda}_g}{(d-1)(d-2)} \,. \tag{3.9}$$

A and B are the normalizations of the nonlinear massless and massive fields $G_{\mu\nu}$ and $M_{\mu\nu}^G$ to be determined later.

Thus, the conclusion is that if a nonlinear theory with PM gauge symmetry exits, its equations of motion on proportional backgrounds must be invariant under the transformations (3.7). This requirement puts constraints on the bimetric theory (2.1) and singles out the potential nonlinear PM cases. The criteria for the invariance are listed below.

1. Recall that for proportional backgrounds $\bar{f} = c^2 \bar{g}$, the bimetric equations of motion imply $\Lambda_g = \Lambda_f$ (2.11), explicitly,

$$\sum_{n=0}^{d-1} {d-1 \choose n} c^n \beta_n = (\alpha c)^{2-d} \sum_{n=1}^d {d-1 \choose n-1} c^n \beta_n.$$
 (3.10)

Generically, this fixes $c = c(\alpha, \beta_i)$ and excludes the possibility of invariance under transformations (3.7) that require changing c. Thus the nonlinear theory can be invariant under (3.7) only for special values of (β_n, α) for which (3.10) leaves c undetermined. This is the necessary condition for the existence of nonlinear PM theories for these (β_n, α) values.

2. When the theory is expressed in term of the massive and massless fields $M_{\mu\nu}^G$ and $G_{\mu\nu}$, the FP mass and the cosmological constant take the form (2.26), or explicitly,

$$\tilde{m}_{\rm FP}^2 = \frac{m^d}{m_g^{d-2}} \frac{(\alpha c)^{2-d}}{A(c)} \sum_{n=1}^{d-1} {d-2 \choose n-1} c^n \beta_n, \qquad (3.11)$$

$$\tilde{\Lambda}_g = \frac{m^d}{m_g^{d-2}} \frac{\left(1 + (\alpha c)^{d-2}\right)^{-1}}{A(c)} \sum_{n=0}^{d-1} {d-1 \choose n} c^n \beta_n.$$
(3.12)

Since the gauge transformations in the nonlinear PM theory involve shifts of c, then $\tilde{m}_{\rm FP}^2$ and $\tilde{\Lambda}_g$, as well as the effective couplings, must become c-independent.

3. The gauge symmetry of the linear PM theory leaves the massless fluctuation $\delta G_{\mu\nu}$ invariant. Hence at the background level too, transformations (3.7) must keep the corresponding nonlinear background $\bar{G}_{\mu\nu}$ invariant. This requirement determines the normalization A(c) of $G_{\mu\nu}$. Indeed from (2.23) and (3.7) one can compute,

$$\Delta_0 \bar{G}_{\mu\nu} = \bar{g}_{\mu\nu} A \left[1 + (\alpha c)^{d-2} \right] \Delta_0(c) \frac{d}{dc} \ln \left(A \left[1 + (\alpha c)^{d-2} \right]^{\frac{d-4}{d-2}} \right) = 0, \quad (3.13)$$

which fixes the normalization, upto a c-independent factor, as,

$$A(c) = \left[1 + (\alpha c)^{d-2}\right]^{\frac{4-d}{d-2}},\tag{3.14}$$

Then, the background massless and massive fields become,

$$\bar{G}_{\mu\nu} = \left[1 + (\alpha c)^{d-2}\right]^{\frac{2}{d-2}} \bar{g}_{\mu\nu} , \qquad \bar{M}_{\mu\nu}^G = 0.$$
 (3.15)

As discussed below, the first criterion is satisfiable only in 3 and 4 dimensions (not counting d < 3 which has trivial dynamics). Then in these cases, the second criterion is satisfied only for the normalizations A(c) determined by the third criterion. Hence, the potential nonlinear PM theories in 3 and 4 dimensions are manifestly invariant under the dS preserving part of the PM gauge transformations. Let us now consider these cases separately.

3.3 Nonlinear PM theory for d=3, 4

In 3 dimensions the condition (3.10) that generically determines the c in $f = c^2 g$ reads

$$(\alpha\beta_0 - \beta_1) + 2(\alpha\beta_1 - \beta_2)c + (\alpha\beta_2 - \beta_3)c^2 = 0.$$
 (3.16)

It leaves c undetermined only when the coefficients of all powers of c vanish,

$$\beta_1 = \alpha \beta_0, \qquad \beta_2 = \alpha \beta_1, \qquad \beta_3 = \alpha \beta_2, \qquad (3.17)$$

or simply, $\beta_n = \alpha^n \beta_0$ for n = 1, 2, 3. Only for these parameter values the restricted gauge transformations (3.7) lead to new proportional backgrounds $f' = c'^2 g'$ with $c' \neq c$, without violating the bimetric equations of motion. For the above parameter values, the mass and cosmological constant (3.11, (3.12) measured in the metric $\bar{G}_{\mu\nu}$ become,

$$\tilde{m}_{\rm FP}^2 = \tilde{\Lambda}_g = \frac{m^3}{m_f} \beta_1 \,. \tag{3.18}$$

Thus they satisfy the Higuchi bound and are indeed independent of c and hence gauge invariant, but only for the normalization $A = [1 + (\alpha c)]$ given by (3.14).

In 4 dimensions (first considered in [17]), the condition (3.10) becomes,

$$\beta_1 + (3\beta_2 - \alpha^2 \beta_0) c + (3\beta_3 - 3\alpha^2 \beta_1) c^2 + (\beta_4 - 3\alpha^2 \beta_2) c^3 + \alpha^2 \beta_3 c^4 = 0.$$
 (3.19)

This leaves c undetermined for

$$\beta_1 = \beta_3 = 0, \qquad \alpha^4 \beta_0 = 3\alpha^2 \beta_2 = \beta_4.$$
 (3.20)

Then the nonlinear PM theory must correspond to this choice of parameters. For these parameters, the mass and cosmological constant satisfy the Higuchi bound,

$$\tilde{m}_{\rm FP}^2 = \frac{2}{3}\tilde{\Lambda}_g = 2\frac{m^4}{m_f^2}\beta_2. \tag{3.21}$$

Again, they are independent of c, and hence gauge invariant, precisely for the normalization A = 1 (3.14) that also renders $\bar{G}_{\mu\nu}$ invariant.

The bimetric action (2.1) with the β 's given above satisfies the necessary conditions for being a nonlinear PM theory for d = 3, 4. The PM gauge symmetry of the nonlinear action (2.1) is not yet known. But the discussion shows that, at least for proportional backgrounds, the nonlinear equations are invariant under the dS preserving subset of the PM gauge transformations (3.7). In fact, this subset can be further extended to the nonlinear dS preserving transformations,

$$c' = c + a$$
, $\bar{g}'_{\mu\nu} = \left[\frac{1 + (\alpha c)^{d-2}}{1 + (\alpha (c+a))^{d-2}} \right]^{\frac{2}{d-2}} \bar{g}_{\mu\nu}$, (3.22)

where a is a finite parameter and the transformation of \bar{f} follows from $\bar{f} = c^2 \bar{g}$. To linear order in $a = \Delta_0(c)$, this reduces to (3.7). The above transformations keep the background $\bar{G}_{\mu\nu}$ (2.23) unchanged and also leave the background $g_{\mu\nu}$ and $f_{\mu\nu}$ equations of motion (2.9) invariant.

3.4 The quadratic action

As a last check, let us consider the quadratic action for the fluctuations $\delta G_{\mu\nu}$ and $\delta M_{\mu\nu}$ given by (2.15),(2.16) and (3.14) with B(c) = 1, that is,

$$\delta G_{\mu\nu} = \left[1 + (\alpha c)^{d-2}\right]^{\frac{4-d}{d-2}} \left(\delta g_{\mu\nu} + c^{d-4} \alpha^{d-2} \delta f_{\mu\nu}\right) , \qquad (3.23)$$

$$\delta M_{\mu\nu} = \frac{1}{2c} \left(\delta f_{\mu\nu} - c^2 \delta g_{\mu\nu} \right) . \tag{3.24}$$

This action is needed to determine the effective Planck masses of $\delta G_{\mu\nu}$ and $\delta M_{\mu\nu}$ and to verify that they are indeed independent of c and hence gauge invariant. Expanding the bimetric action (2.1) to second order and expressing the result in terms of the above combinations one obtains,

$$S^{(2)} = m_g^{d-2} \int d^d x \sqrt{\bar{G}} \left[-\delta G \tilde{\mathcal{E}} \delta G + \frac{\tilde{\Lambda}_g}{(d-2)} \left(\text{Tr}[\delta G^2] - \frac{1}{2} \text{Tr}[\delta G]^2 \right) - \delta M \tilde{\mathcal{E}} \delta M - \frac{\tilde{\Lambda}_g}{(d-2)} \left(\text{Tr}[\delta M^2] - \frac{1}{2} \text{Tr}[\delta M]^2 \right) + \frac{\tilde{m}_{\text{FP}}^2}{4} \left(\text{Tr}[\delta M]^2 - \text{Tr}[\delta M^2] \right) \right]. \quad (3.25)$$

Here, \tilde{m}_{FP}^2 and $\tilde{\Lambda}_g$ are given by (3.11), (3.12) which become independent of c for the PM parameter values. Also note that the Planck masses are independent of c for the correct normalization in (3.23) determined by other considerations. Furthermore, the quadratic action fixes B(c) = 1 since otherwise, the δM sector will have a c-dependent coupling.

In the appendix we outline the structure of the complete nonlinear action expressed in terms of $G_{\mu\nu}$ and $M_{\mu\nu}^G$. This is useful for computing the cubic and higher order interactions of the PM field to be compared with the explicit cubic vertex computations in [12, 14, 16]. We do not attempt such a comparison here.

3.5 Absence of nonlinear PM theory in d>4

The necessary condition for the existence of nonlinear PM theories is that (3.10) leaves c undetermined. On relabeling the sum on the right hand side, this equation can be recast as,

$$\left(\alpha^{d-2} \sum_{n=0}^{d-1} A_n \,\beta_n - \sum_{n=3-d}^2 B_{n+d-2} \,\beta_{n+d-2}\right) \, c^n = 0 \,, \tag{3.26}$$

where,

$$A_n = \begin{pmatrix} d-1 \\ n \end{pmatrix}, \qquad B_n = \begin{pmatrix} d-1 \\ n-1 \end{pmatrix}. \tag{3.27}$$

To leave c undetermined, the coefficients of each power of c must vanish separately. For d > 4, this gives the following 3 sets of equations:

$$3 - d \le n \le -1: \quad \beta_{n+d-2} = 0, \tag{3.28}$$

$$0 \le n \le 2: \quad \alpha^{d-2} A_n \,\beta_n - B_{n+d-2} \,\beta_{n+d-2} = 0, \tag{3.29}$$

$$3 \le n \le d - 1: \quad \beta_n = 0.$$
 (3.30)

The first set implies that $\beta_m = 0$ for $1 \le m \le d - 3$ whereas the last set implies that $\beta_m = 0$ for $3 \le m \le d - 1$. Then the second set implies $\beta_0 = \beta_d = 0$. Hence, $\beta_n = 0$ for $n = 0, \dots, d$.

To recapitulate, the bimetric action (2.1), when linearized around proportional backgrounds, leads to the massive FP equation (2.25). On the Higuchi bound (3.1), this gives a linear PM theory in any dimension. The consistency of the dS-preserving part of the linear PM gauge symmetry with dynamical backgrounds provided the necessary (bot not sufficient) condition for the existence of nonlinear PM theories. This criterion was satisfied in 3 and 4 dimensions but not for d > 4.

4 Partial Masslessness with Lanczos-Lovelock terms

Our nonlinear results are consistent with the observation in [12] that in a 2-derivative theory, cubic spin-2 interactions with PM gauge symmetry exist only in 3 and 4 dimensions. However, [12] also finds that in the presence of higher derivative terms, cubic PM interactions exist even for d > 4. These higher derivative terms reduce to 2-derivative terms in d = 4. This hints that the nonlinear higher derivative interactions correspond to Lanczos-Lovelock (LL) terms which will also preserve the unitarity of the bimetric theory. This section considers the LL extension of the bimetric theory from the point of view of the PM analysis. Such a theory has been considered in [50] in a different context. Recently, in [51] the spectrum of Einstein-Hilbert gravity with LL terms was studied in dS backgrounds. Here, this analysis is generalized to the bimetric theory and used to investigate the existence of PM theories beyond d = 4.

To summarize, it is shown that, for $\bar{f}=c^2\bar{g}$ and around dS backgrounds, the structure of massless and massive fluctuations in bimetric theory with LL terms is exactly the same as in pure bimetric theory, though with modified parameters. Hence the PM analysis previously developed for the pure bimetric case also applies here. In particular, the linearized theory has PM gauge symmetry on the Higuchi bound. Then for a nonlinear extension of the theory to exist, the necessary condition is that the equations of motion must leave c undetermined. This condition is easily implemented if one obtains a polynomial equation for c.

For $d \leq 4$ the LL terms are of no consequence and the pure bimetric analysis goes through. The first new contribution is from the quadratic LL term for $d \geq 5$ and this remains the only contribution for d = 5, 6. In this case, one can easily obtain a polynomial equation for c and determine the potential PM theory. The cubic LL term starts contributing for d > 6 but in this case one does not automatically obtain a polynomial equation for c. However, using a decomposition into the Gröbner basis one again obtains a polynomial equation for c, as discussed in subsection 4.3. The analytical derivation of the PM theory in d = 5 is presented in subsection 4.4, whereas the PM theories for d = 6, 7, 8 are determined with the help of a computer and are discussed in subsection 4.5. Beyond this, the analysis gets more involved and is not attempted here.

4.1 Bimetric theory with Lanczos-Lovelock terms in dS spacetimes

Adding higher order LL invariants $\mathcal{L}_{(n)}$ for both $g_{\mu\nu}$ and $f_{\mu\nu}$ to the bimetric action gives,

$$S = m_g^{d-2} \int d^d x \sqrt{g} \left[R(g) + \sum_{n=2}^{[d/2]} l_n^g \mathcal{L}_{(n)}(g) \right] + m_f^{d-2} \int d^d x \sqrt{f} \left[R(f) + \sum_{n=2}^{[d/2]} l_n^f \mathcal{L}_{(n)}(f) \right] - 2m^d \int d^d x \sqrt{g} V\left(\sqrt{g^{-1}f}; \beta_n\right),$$
(4.1)

where l_n^g and l_n^f are coupling constants of mass dimension -2(n-1). The sum terminates at [d/2] (integer part of) since the Lovelock terms vanish identically when d < 2n and are topological invariants when d = 2n. Our conventions for the Lovelock invariants are,

$$\mathcal{L}_{(n)} = \frac{1}{2^n} \delta^{\mu_1 \nu_1 \dots \mu_n \nu_n}_{\alpha_1 \beta_1 \dots \alpha_n \beta_n} \prod_{r=1}^n R^{\alpha_r \beta_r}_{\mu_r \nu_r}, \quad \delta^{\mu_1 \nu_1 \dots \mu_n \nu_n}_{\alpha_1 \beta_1 \dots \alpha_n \beta_n} = \frac{1}{n!} \delta^{\mu_1}_{[\alpha_1} \delta^{\nu_1}_{\beta_1} \dots \delta^{\mu_n}_{\alpha_n} \delta^{\nu_1}_{\beta_1]}. \tag{4.2}$$

Note that the action (4.1) is invariant under simultaneous interchange of,

$$\alpha^{\frac{2-d}{2}}g_{\mu\nu}\longleftrightarrow \alpha^{\frac{d-2}{2}}f_{\mu\nu}, \qquad \beta_n\longrightarrow \alpha^{2n-d}\beta_{d-n}, \qquad l_n^f\longleftrightarrow \alpha^{2-2n}l_n^g.$$
 (4.3)

The equations of motion obtained from this action read,

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) + \sum_{n=2}^{[d/2]} l_n^g \mathcal{G}_{\mu\nu}^{(n)}(g) + \frac{m^d}{m_g^{d-2}} V_{\mu\nu}^g = 0, \qquad (4.4)$$

$$R_{\mu\nu}(f) - \frac{1}{2}g_{\mu\nu}R(f) + \sum_{n=2}^{[d/2]} l_n^f \mathcal{G}_{\mu\nu}^{(n)}(f) + \frac{m^d}{m_f^{d-2}} V_{\mu\nu}^f = 0, \qquad (4.5)$$

where the Lovelock tensors $\mathcal{G}_{\mu\nu}^{(n)}$ appear as the result of varying the Lovelock invariants.

We are interested in the maximally symmetric solutions of the above equations that allow for massive spin-2 excitations. In the bimetric setup, these belong to the class of proportional backgrounds $f_{\mu\nu} = c^2 g_{\mu\nu}$. For a maximally symmetric spacetime with cosmological constant λ , the curvatures are,

$$R_{\mu\nu\rho\sigma}(g) = \frac{2\lambda}{(d-1)(d-2)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma} \right) , \quad R_{\mu\nu}(g) = \frac{2\lambda}{d-2} g_{\mu\nu} , \quad R(g) = \frac{2d\lambda}{d-2} , \quad (4.6)$$

with the corresponding equations for $f_{\mu\nu}$. With this ansatz it is enough to consider the traces of the equations of motion (4.4) and (4.5) to obtain two equations that generically determine λ and c in terms of the parameters of the theory. The computation is simplified by,

$$g^{\mu\nu}\mathcal{G}^{(n)}_{\mu\nu}(g) = \frac{2n-d}{2}\mathcal{L}_{(n)}(g), \qquad (4.7)$$

which allows us to work with the Lovelock invariants, rather than the corresponding Lovelock tensors. Furthermore, for the proportional ansatz we have,

$$\mathcal{L}_{(n)}(f) = c^{-2n} \mathcal{L}_{(n)}(g). \tag{4.8}$$

Now, for the ansatz (4.6), the Lovelock invariants become,

$$\mathcal{L}_{(n)} = N_n(d)\lambda^n$$
, with $N_n(d) = \frac{2^n d!}{(d-1)^n (d-2)^n (d-2n)!}$. (4.9)

Using the above relations, the traced equations of motion (4.4) and (4.5) give,

$$\lambda + \sum_{n=2}^{[d/2]} l_n^g \frac{d - 2n}{2d} N_n(d) \lambda^n - \Lambda_g = 0, \qquad (4.10)$$

$$\lambda + \sum_{n=2}^{[d/2]} c^{2-2n} l_n^f \frac{d-2n}{2d} N_n(d) \lambda^n - \Lambda_f = 0.$$
 (4.11)

The contributions $\Lambda_{g,f}$ from the bimetric potential are still given by (2.10). In general, these equations determine the cosmological constant λ and the proportionality constant c. Hence, they specify the maximally symmetric solutions in bimetric theory with LL terms.

Now, we consider the spectrum of fluctuations in the above dS backgrounds. The analysis is greatly simplified using results of [51], which considered Einstein-Hilbert gravity with LL terms, S_{EH+LL} . For a single metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ described by S_{EH+LL} , [51] showed that the quadratic action for $\delta g_{\mu\nu}$ in an (A)dS background $\bar{g}_{\mu\nu}$ was exactly the same as the quadratic action in pure Einstein-Hilbert gravity in the same background but with a modified, effective, Planck mass. Applied to the bimetric action with LL terms (4.1) this implies that at the quadratic level, the theory has the same structure as the pure bimetric theory (2.1) discussed earlier, but now with modified Planck masses,

$$\bar{m}_g^{d-2} = m_g^{d-2} \left[1 + (d-3)! \sum_{n=2}^{[d/2]} \frac{n(d-2n)}{(d-2n)!} \left(\frac{2\lambda}{(d-1)(d-2)} \right)^{n-1} l_n^g \right], \tag{4.12}$$

$$\bar{m}_f^{d-2} = m_f^{d-2} \left[1 + (d-3)! \sum_{n=2}^{[d/2]} \frac{n(d-2n)}{(d-2n)!} \left(\frac{2\lambda}{(d-1)(d-2)} \right)^{n-1} c^{2-2n} l_n^f \right]. \tag{4.13}$$

This implies that, just as before, the spectrum of fluctuations consists of massive and massless spin-2 modes in a dS background with cosmological constant λ . In particular, the Fierz-Pauli mass has the same form as in (2.20), but now involves the modified Planck masses,

$$m_{\rm FP}^2 = \frac{m^d}{\bar{m}_g^{d-2}} \left(1 + \left(c \frac{\bar{m}_f}{\bar{m}_g} \right)^{2-d} \right) \sum_{k=1}^{d-1} {d-2 \choose k-1} c^k \beta_k. \tag{4.14}$$

Since the massive mode satisfies the FP equation, it is obvious that the linear theory will exhibit partial masslessness at the Higuchi bound, $m_{\text{FP}}^2 = \frac{2}{d-1}\lambda$. Below we show that in the presence of the LL terms, nonlinear potentially PM theories can be found even for d > 4. Before this, let us look at the quadratic and quartic LL terms in more detail.

4.2 The quadratic Lanczos-Lovelock term

We now apply the general results of the previous section to the quadratic LL term which is the Gauss-Bonnet combination,

$$\mathcal{L}_{(2)} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{4.15}$$

This contribution is trivial for $d \leq 4$ but modifies the bimetric theory for d > 4. For a maximally symmetric spacetime with cosmological constant λ it reduces to

$$\mathcal{L}_{(2)} = \frac{4d(d-3)}{(d-1)(d-2)}\lambda^2. \tag{4.16}$$

Then the traced bimetric equations of motion on maximally symmetric, proportional backgrounds become

$$\lambda + \frac{2(d-3)(d-4)}{(d-1)(d-2)} l_2^g \lambda^2 - \Lambda_g = 0, \qquad (4.17)$$

$$\lambda + \frac{2(d-3)(d-4)}{(d-1)(d-2)}c^{-2}l_2^f\lambda^2 - \Lambda_f = 0, \qquad (4.18)$$

The above equations determine λ and c in terms of the parameters of theory. For the purpose of PM analysis, we prefer that c is determined by a polynomial equation. For this, the solution for λ can be easily obtained in the appropriate form on multiplying the f-equation by $c^2 l_2^g / l_2^f$ and subtracting it from the g-equation. This gives,

$$\lambda = \frac{c^2 l_2^g \Lambda_f - l_2^f \Lambda_g}{c^2 l_2^g - l_2^f} \,. \tag{4.19}$$

Then plugging this expression into the difference of (4.17) and (4.18) gives

$$\frac{2(d-3)(d-4)}{(d-1)(d-2)}l_2^g(l\Lambda_g - c^2\Lambda_f)^2 + (c^4 - lc^2)(\Lambda_f - \Lambda_g) = 0,$$
(4.20)

where $l \equiv l_2^f/l_2^g$. This is the desired polynomial equation for c which is a generalization of the previously considered condition $\Lambda_f = \Lambda_g$. In order to arrive at the PM candidate, we will demand that this equation leaves c undetermined.

The effective Planck masses in terms of which the FP mass should be expressed now read,

$$\bar{m}_g^{d-2} = m_g^{d-2} \left(1 + \frac{4(d-3)(d-4)}{(d-1)(d-2)} l_2^g \lambda \right), \qquad \bar{m}_f^{d-2} = m_f^{d-2} \left(1 + \frac{4(d-3)(d-4)}{(d-1)(d-2)} c^{-2} l_2^f \lambda \right). \tag{4.21}$$

These results are sufficient to investigate the most general PM theories in dimension d = 5 and d = 6 where only the quadratic LL term can be added to the bimetric action.

4.3 The cubic Lanczos-Lovelock term

The cubic LL invariant is given by

$$\mathcal{L}_{(3)} = R^{3} - 8R^{\mu\nu\rho\sigma}R_{\mu\ \rho}^{\ \tau}{}^{\gamma}R_{\nu\tau\sigma\gamma} + 4R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\ \tau\gamma}R_{\rho\sigma\tau\gamma} - 24R^{\mu\nu}R^{\rho\sigma\tau}{}_{\mu}R_{\rho\sigma\tau\nu} + 3RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 24R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho\nu\sigma} + 16R^{\mu\nu}R^{\rho}{}_{\mu}R_{\nu\rho} - 12RR_{\mu\nu}R^{\mu\nu},$$
(4.22)

which on maximally symmetric backgrounds reduces to

$$\mathcal{L}_{(3)} = \frac{8d(d-3)(d-4)(d-5)}{(d-1)^2(d-2)^2} \lambda^3.$$
 (4.23)

The trace of the equations of motion on proportional, maximally symmetric backgrounds then gives,

$$\lambda + \frac{2(d-3)(d-4)}{(d-1)(d-2)}l_2^g\lambda^2 + \frac{4(d-3)(d-4)(d-5)(d-6)}{(d-1)^2(d-2)^2}l_3^g\lambda^3 - \Lambda_g = 0, \quad (4.24)$$

$$\lambda + \frac{2(d-3)(d-4)}{(d-1)(d-2)}c^{-2}l_2^f\lambda^2 + \frac{4(d-3)(d-4)(d-5)(d-6)}{(d-1)^2(d-2)^2}c^{-4}l_3^f\lambda^3 - \Lambda_f = 0.$$
 (4.25)

Subtracting off the cubic term will result in a quadratic equation for λ and hence the solution will involve a square root. However, in order to apply our method for determining the PM candidate, we need to arrive at a polynomial equation in c. It turns out that this can still be achieved by decomposing the equations into a Gröbner basis of equivalent polynomials instead. In the cases we have been able to reduce in this way it turns out that we then get a linear equation for λ together with a consistency condition on the coefficients of the original polynomials which can be converted into a polynomial equation in c. Here, for simplicity, we present the results for the Gröbner reduction after having imposed the symmetry relation $l_n^f = \alpha^{2-2n} l_n^g$ which also covers the case of PM theories (although later the PM analysis is performed starting with the most general set of parameters). We then obtain a linear equation for λ with the solution

$$\lambda = \frac{n_3(1-q^2)(1+q^2)^2 \Delta_q \Lambda - n_2^2 q^2 (1-q^2) [\Delta_q \Lambda + 2q^2 (\Lambda_g - \Lambda_f)] - n_2 n_3 (\Delta_q \Lambda)^2}{(1-q^2)[n_3(1-q^2)(1+q^2)^3 - n_2^2 q^2 (1-q^4 - n_2(\Lambda_g - q^2\Lambda_f))]}, \quad (4.26)$$

where, we have used the following notion,

$$\Delta_q \Lambda = \Lambda_g - q^4 \Lambda_f, \quad q = \alpha c, \quad n_k = \frac{d - 2n}{2d} N_k(d) l_k^g.$$
(4.27)

A consistency condition on the coefficients of the original polynomials is,

$$n_3^2(\Lambda_g - q^4\Lambda_f)^3 + n_2^2q^4(1 - q^2) \left[q^4(\Lambda_g - \Lambda_f - n_2\Lambda_f^2) - n_2\Lambda_g^2 - q^2(\Lambda_g - \Lambda_f - 2n_2\Lambda_g\Lambda_f) \right] - n_3q^2(1 - q^2) \left[q^6(1 + 3n_2\Lambda_f)(\Lambda_g - \Lambda_f) + q^8(\Lambda_g - \Lambda_f - n_2\Lambda_f^2) - n_2\Lambda_g^2 \right] - q^2(1 + 3n_2\Lambda_g)(\Lambda_g - \Lambda_f) - q^4(\Lambda_g - \Lambda_f - 2n_2\Lambda_g\Lambda_f) = 0.$$
 (4.28)

On multiplying by a factor $c^{3(d-7)}$, one obtains a polynomial equation in c. The requirement that this leaves c undetermined constrains the β_n and the l_n^f as given in Table 1. With these results for the cubic term we can investigate PM theories up to d=8 in full generality.

4.4 Partial Masslessness with LL terms in d = 5

Let us now consider the nonlinear PM candidate with higher derivative interactions in five dimensions where only the quadratic LL term is non-vanishing. This theory is parametrized by the six β_n , along with l_2^g , l_2^f and the two Planck masses m_g , m_f . In this case, (2.10) gives,

$$\Lambda_g = \frac{m^5}{m_g^3} (\beta_0 + 4c\beta_1 + 6c^2\beta_2 + 4c^3\beta_3 + c^4\beta_4), \qquad (4.29)$$

$$\Lambda_f = \frac{m^5}{m_a^3} (\alpha c)^{-3} (c\beta_1 + 4c^2\beta_2 + 6c^3\beta_3 + 4c^4\beta_4 + c^5\beta_5).$$
 (4.30)

Inserting these into (4.20) gives a polynomial with nine different powers in c. Demanding it to be independent of c by setting the coefficient of each power of c to zero therefore gives nine conditions on the parameters that can be explicitly solved.

To simplify the presentation, here we first impose the extra requirement that the parameters of the PM theory must correspond to a fixed point of the interchange symmetry (4.3), or else one ends up with two sets of nonlinear PM theories related by (4.3). Of course, we have checked that the unique nontrivial solution is also obtained without making this ansatz. Then this requirement gives,

$$\beta_n = \alpha^{2n-d} \beta_{d-n} \,, \qquad l_n^f = \alpha^{2-2n} l_n^g \,,$$
 (4.31)

or, explicitly, $\beta_5 = \alpha^5 \beta_0$, $\beta_4 = \alpha^3 \beta_1$, $\beta_3 = \alpha \beta_2$, $l_2^f = \alpha^{-2} l_2^g$. Imposing these, it follows that the nine equations reduce to three independent ones,

$$(\alpha\beta_0 - \beta_1)^2 - 3\kappa\alpha\beta_1 = 0, \qquad (4.32)$$

$$2(\alpha\beta_0 - \beta_1)(\alpha\beta_1 - \beta_2) - 3\kappa\alpha\beta_2 = 0, \qquad (4.33)$$

$$16(\alpha\beta_1 - \beta_2)^2 - 18\kappa\alpha^2\beta_2 + 3\kappa\alpha^4\beta_0 + 3\kappa\alpha^3\beta_1 = 0, \qquad (4.34)$$

where $\kappa \equiv \frac{m_g^3}{l_g^2 m^5}$ is a dimensionless parameter. The structure suggests the following definitions,

$$\beta_0 \equiv -3\kappa b_0, \qquad \beta_1 \equiv -3\kappa \alpha b_1, \qquad \beta_2 \equiv -3\kappa \alpha^2 b_2.$$
 (4.35)

Then the system reduces to the three equations,

$$(b_0 - b_1)^2 + b_1 = 0$$
, $2(b_0 - b_1)(b_1 - b_2) + b_2 = 0$, $16(b_1 - b_2)^2 - b_0 - b_1 + 6b_2 = 0$, (4.36) which are solved by $b_0 = \frac{5}{36}$, $b_1 = -\frac{1}{36}$, $b_2 = \frac{1}{72}$, or,

$$\beta_{0} = -\frac{5}{12}\kappa, \quad \beta_{1} = \frac{\alpha}{12}\kappa, \quad \beta_{2} = -\frac{\alpha^{2}}{24}\kappa, \quad \beta_{3} = -\frac{\alpha^{3}}{24}\kappa,$$

$$\beta_{4} = \frac{\alpha^{4}}{12}\kappa, \quad \beta_{5} = -\frac{5\alpha^{5}}{12}\kappa, \quad l_{2}^{f} = \alpha^{-2}l_{2}^{g}.$$
(4.37)

These determine all the β_n and l_2^f in terms of l_2^g , m_g and m_f . Inserting these into the solution for λ in equation (4.19), and into the FP mass (4.14) gives the linear Higuchi bound,

$$m_{\rm FP}^2 = \frac{1}{4l_2^g} (-1 + \alpha c - \alpha^2 c^2) = \frac{1}{2} \lambda.$$
 (4.38)

4.5 Partial masslessness with LL terms in d > 5

In d = 6, the cubic LL term is topological so it is still enough to consider the quadratic term and the solution for the PM parameters can be obtained in a straightforward way. Up to a sign ambiguity in β_3 , all β_n are uniquely fixed in terms of l_2^g .

In d=7, we also add the cubic Lovelock term and use the results of section 4.3. It turns out that there exists no nontrivial solution to the constraint equations on the β_n . Hence there is no PM theory in seven dimensions even with LL terms.

In d = 8 it is still sufficient to consider only the quadratic and cubic terms. Remarkably, the resulting PM candidate theory has two free parameters. In particular, it is possible to set the cubic coupling to zero and obtain a PM theory with only the quadratic Lovelock term.

For d = 9, 10 it has been checked that adding both $\mathcal{L}_{(2)}$ and $\mathcal{L}_{(3)}$, or adding $\mathcal{L}_{(4)}$ alone, does not lead to nontrivial solutions for the parameters. The only possibility then is to add all terms up to quartic order, which we have not been able to investigate with our methods at hand. Hence we can only make definite statements about dimensions up to d = 8.

The results for the PM parameter values are summarized in Table 1, where, to simplify the presentation, we define

$$\kappa_d^{(2)} \equiv \frac{m_g^{d-2}}{l_2^g m^d}, \quad \kappa_d^{(3)} \equiv \frac{l_3^g m_g^{d-2}}{(l_2^g)^3 m^d}.$$
(4.39)

$\kappa_{d} = \frac{1}{l_2^g m^d},$	$\kappa_d = \frac{(l_2^g)^3 m^d}{(l_2^g)^3 m^d}.$	(4.39)

dimension	3	4	5	6	8
l_2^g	-	-	l_2^g	l_2^g	l_2^g
l_2^f	-	-	$\alpha^{-2}l_2^g$	$\alpha^{-2}l_2^g$	$\alpha^{-2}l_2^g$
l_3^g	-	-	-	-	l_3^g
l_3^f	-	-	-	-	$\alpha^{-4}l_3^g$
β_0	β_0	β_0	$-\frac{5}{12}\kappa_{5}^{(2)}$	$-\frac{5}{12}\kappa_6^{(2)}$	$-\frac{21}{1600}\left(20\kappa_8^{(2)}+3\kappa_8^{(3)}\right)$
β_1	$\alpha\beta_0$	0	$\frac{\alpha}{12}\kappa_5^{(2)}$	0	0
β_2	$\alpha^2 \beta_0$	$\frac{\alpha^2}{3}\beta_0$	$-\frac{\alpha^2}{24}\kappa_5^{(2)}$	$\frac{\alpha^2}{12}\kappa_6^{(2)}$	$-\frac{9\alpha^2}{1600}\kappa_8^{(3)}$
β_3	$\alpha^3 \beta_0$	0	$ \frac{\frac{\alpha}{12}\kappa_5^{(2)}}{-\frac{\alpha^2}{24}\kappa_5^{(2)}} \\ -\frac{\alpha^3}{24}\kappa_5^{(2)} $	$\pm \frac{\alpha^3}{6\sqrt{2}} \kappa_6^{(2)}$	0
β_4	-	$\alpha^4 \beta_0$	$\frac{\alpha^4}{12}\kappa_5^{(2)}$	$\frac{\alpha^4}{12}\kappa_6^{(2)}$	$\frac{3\alpha^4}{8000} \left(20\kappa_8^{(2)} - 9\kappa_8^{(3)} \right)$
β_5	-	-	$-\frac{5\alpha^5}{12}\kappa_5^{(2)}$	0	0
β_6	-	-	-	$-\frac{5\alpha^{6}}{12}\kappa_{6}^{(2)}$	$-\frac{9\alpha^{6}}{1600}\kappa_{8}^{(3)}$
β_7	-	-	-	-	0
β_8	_	-	-	-	$-\frac{21\alpha^8}{1600} \left(20\kappa_8^{(2)} + 3\kappa_8^{(3)}\right)$

 Table 1. Parameters of the PM candidates

Note that in all cases the parameters satisfy the symmetry property (4.31) and the linear Higuchi bound,

$$d = 5: m_{\text{FP}}^2 = -\frac{1}{4l_2^g} \left(1 - \alpha c + (\alpha c)^2 \right) = \frac{\lambda}{2},$$

$$d = 6: m_{\text{FP}}^2 = -\frac{1}{3l_2^g} \left(1 \pm \sqrt{2}\alpha c + (\alpha c)^2 \right) = \frac{2\lambda}{5},$$

$$d = 8: m_{\text{FP}}^2 = -\frac{3}{20l_2^g} \left(1 + (\alpha c)^2 \right) = \frac{2\lambda}{7}.$$

$$(4.40)$$

5 Discussion

Our results are summarized in the introduction. One of the issues left unanswered in the present analysis is the actual form of the gauge transformation in the nonlinear PM theory. Currently one can see the nonlinear form of this gauge transformation in the nonlinear theory in dS backgrounds only for constant gauge parameters. Non-constant gauge parameters will move the background away from dS. However, the analysis of cubic interactions [12, 14, 16] shows that, to cubic order, the theory is invariant under the full linear transformation, and not just its constant part employed in this paper. Another evidence for the existence of a nonlinear symmetry comes from the study of cosmological solutions in [36] based on a FLRW-type ansatz. It turns out that for the PM values of parameters (3.20) and the cosmological ansatz of [36], the equations of motion leave a complete function of time undetermined which indicates a nonlinear gauge invariance. Furthermore, using a different approach, the authors in [15] consider partial masslessness in massive gravity in a certain decoupling limit and find a nonlinear symmetry that exists in that limit. This analysis is not directly comparable to construction here, but could be taken as another indication of the presence of the nonlinear symmetry.

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A Mathematical details

Here we provide the details for deriving the equations of motion from the bimetric action (2.1), computing the cosmological constants on proportional backgrounds and finding the Fierz-Pauli mass of the linear fluctuations.

A.1 Bimetric equations of motion

The bimetric interaction potential in (2.1) is given in terms of $e_n(S)$ where $S = \sqrt{g^{-1}f}$. For a $d \times d$ matrix S, one has

$$\det(\mathbb{1} + \lambda S) = \sum_{n=0}^{d} \lambda^n e_n(S). \tag{A.1}$$

 $e_n(S)$ are the elementary symmetric polynomials of the eigenvalues of of S and can be iteratively constructed starting with $e_0(S) = 1$ and using the Newton's identities,

$$e_n(S) = -\frac{1}{n} \sum_{k=1}^n (-1)^k \operatorname{Tr}(S^k) e_{n-k}(S).$$
(A.2)

In d dimensions, the iteration ends with $e_d(S) = \det S$ and then $e_n(S) = 0$ for n > d. Obviously, $e_n(\lambda S) = \lambda^n e_n(S)$. To obtain the equations of motion, one needs the variations,

$$\delta e_n(S) = -\sum_{m=1}^n (-1)^m \text{Tr}(S^{m-1} \delta S) e_{n-m}(S).$$
(A.3)

These follow from (A.2). Hence,

$$\delta[\det(S^{-1}) e_n(S)] = -\det(S^{-1}) \sum_{m=0}^{n} (-1)^m \operatorname{Tr}(S^{m-1} \delta S) e_{n-m}(S).$$
 (A.4)

If S is the square-root of a matrix $E, S^2 = E$, then

$$Tr(S^{m-1}\delta S) = \frac{1}{2}Tr(\sqrt{E}^{m-2}\delta E)$$
(A.5)

Note that for matrices, this relation holds only under the trace. If $E = g^{-1}f$, then on varying g^{-1} , changing the summation variable from m to r = n - m, and cancelling $\sqrt{|f|}$, one gets,

$$\frac{\delta}{\delta g^{\mu\nu}} \left[\sqrt{|g|} \, e_n(S) \right] = -\frac{1}{2} \sqrt{|g|} (-1)^n \, g_{\mu\lambda} \, Y_{(n)\nu}^{\lambda}(S) \,, \tag{A.6}$$

where, the matrices $Y^{\mu}_{(n)\nu}(S)$ are given by,

$$Y_{(n)}(S) = \sum_{k=0}^{n} (-1)^k S^{n-k} e_k(S).$$
(A.7)

Using (A.6), for V given by (2.3) and satisfying the property (2.5), it is now straightforward to compute,

$$V_{\mu\nu}^{g} = \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \left[-2\sqrt{|g|}V \right] = g_{\mu\lambda} \sum_{n=0}^{d-1} (-1)^{n} \beta_{n} Y_{(n)\nu}^{\lambda}(\sqrt{g^{-1}f}), \qquad (A.8)$$

$$V_{\mu\nu}^{f} = \frac{1}{\sqrt{|f|}} \frac{\delta}{\delta f^{\mu\nu}} \left[-2\sqrt{|f|}V \right] = f_{\mu\lambda} \sum_{n=0}^{d-1} (-1)^{n} \beta_{d-n} Y_{(n)\nu}^{\lambda}(\sqrt{f^{-1}g}).$$
 (A.9)

This leads to the bimetric equations of motion (2.6). Note that the two equations are obtainable from each other by the interchanges $g_{\mu\nu} \leftrightarrow f_{\mu\nu}$, $\beta_n \leftrightarrow \beta_{d-n}$, $m_g \leftrightarrow m_f$.

A.2 Proportional backgrounds and the cosmological constants

Now consider the equations of motion (2.6) for the background ansatz $\bar{f}_{\mu\nu} = c^2 \bar{g}_{\mu\nu}$, implying $\bar{S} = c\mathbb{1}$. It is obvious that the potential terms (A.8), (A.9) become cosmological contributions,

$$\frac{m^d}{m_g^{d-2}} \bar{V}_{\mu\nu}^g = \bar{g}_{\mu\nu} \Lambda_g, \qquad \frac{m^d}{m_f^{d-2}} \bar{V}_{\mu\nu}^f = \bar{g}_{\mu\nu} \Lambda_f.$$
(A.10)

 Λ_g and Λ_f are expressed in terms of $Y_{(n)}(c\mathbb{1})$. To compute this, consider (A.1) for $\bar{S} = c\mathbb{1}$. Since $\det(1 + \lambda \bar{S}) = (1 + \lambda c)^d$ and $e_k(c\mathbb{1}) = c^k e_k(\mathbb{1})$, it follows that

$$e_k(\mathbb{1}) = \binom{d}{k} \equiv \frac{d!}{k!(d-k)!}, \qquad \sum_{k=0}^n (-1)^k e_k(\mathbb{1}) = \sum_{k=0}^n (-1)^k \binom{d}{k} = (-1)^n \binom{d-1}{n}.$$
 (A.11)

Exactly this sum appears in $Y_{(n)}(c1)$ (A.7) which becomes,

$$Y_{(n)}(c\mathbb{1}) = (-1)^n c^n \binom{d-1}{n}, \qquad Y_{(n)}(c^{-1}\mathbb{1}) = (-1)^n c^{-n} \binom{d-1}{n}.$$
 (A.12)

On substituting in $\bar{V}_{\mu\nu}^{g,f}$ one reads off the cosmological constants Λ_g and Λ_f from (A.10) as,

$$\Lambda_g = \frac{m^d}{m_g^{d-2}} \sum_{n=0}^{d-1} {d-1 \choose n} c^n \beta_n, \qquad \Lambda_f = \frac{m^d}{m_f^{d-2}} c^{2-d} \sum_{n=1}^d {d-1 \choose n-1} c^n \beta_n.$$
 (A.13)

Therefore, for proportional backgrounds, the bimetric equations reduce to two copies of Einstein's equations (2.9). The consistency of these background equations with each other then requires $\Lambda_g = \Lambda_f$, which generically determines c.

A.3 Fluctuations and the Fierz-Pauli mass

Let us now consider linear perturbations around the proportional backgrounds (2.8),

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,, \quad f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu} \,,$$
 (A.14)

with $\bar{f} = c^2 \bar{g}$. The corresponding linearized equations obtained from (2.6) are,

$$\delta \left(R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) \right) + \frac{m^d}{m_g^{d-2}} \delta V_{\mu\nu}^g = 0, \qquad \delta \left(R_{\mu\nu}(f) - \frac{1}{2} f_{\mu\nu} R(f) \right) + \frac{m^d}{m_f^{d-2}} \delta V_{\mu\nu}^f = 0.$$
(A.15)

To compute these explicitly, we need,

$$\delta V_{\mu\nu}^{g}|_{\bar{S}} = \frac{m_g^{d-2}}{m^d} \Lambda_g \, \delta g_{\mu\nu} + \bar{g}_{\mu\lambda} \sum_{n=1}^{d-1} (-1)^n \beta_n \, \delta Y_{(n)\nu}^{\lambda}(S)|_{\bar{S}}$$
(A.16)

$$\delta V_{\mu\nu}^f \Big|_{\bar{S}} = \frac{m_f^{d-2}}{m^d} \Lambda_f \, c^{-2} \delta f_{\mu\lambda} + c^2 \bar{g}_{\mu\lambda} \sum_{n=1}^{d-1} (-1)^n \beta_{d-n} \, \delta Y_{(n)\nu}^{\lambda}(S^{-1}) \Big|_{\bar{S}} \,, \tag{A.17}$$

The first term, obviously, is the contribution of the cosmological constants to the linearized fluctuation equations. The second term contains the FP mass and involve δg and δf only

in the combination $\delta S^{\mu}_{\ \nu} = \frac{1}{2c} \bar{g}^{\mu\lambda} (\delta f - c^2 \delta g)_{\lambda\nu}$. Hence the massive mode must contain this combination of the fields. Since $\delta Y_0 = 0$, the mass term is independent of both β_0 and β_d . From (A.7),

$$\delta Y_{(n)}(S)|_{\bar{S}} = \sum_{r=0}^{n} (-1)^r \left[(n-r)c^{n-1} e_r(1) \delta S + c^{n-r} \delta e_r|_{\bar{S}} \right]. \tag{A.18}$$

The first term in $\delta Y_{(n)}$, proportional to δS , involves,

$$n\sum_{r=0}^{n}(-1)^{r}e_{r}(1) = n(-1)^{n}\binom{d-1}{n} = (d-1)(-1)^{n}\binom{d-2}{n-1}$$
(A.19)

$$\sum_{r=1}^{n} (-1)^r r \, e_r(1) = d \, (-1)^n \binom{d-2}{n-1} \tag{A.20}$$

To compute the second term in $\delta Y_{(n)}|$, proportional to $\text{Tr}(\delta S)$, note that (A.3) and (A.11) give,

$$\delta e_r |_{\bar{S}} = -c^{r-1} \operatorname{Tr}(\delta S) \sum_{m=1}^r (-1)^m e_{r-m}(\mathbb{1}) = c^{r-1} \operatorname{Tr}(\delta S) \begin{pmatrix} d-1 \\ r-1 \end{pmatrix}.$$
 (A.21)

with $\delta e_0 = 0$. Then, the second term in (A.18) contains, using the second equation in (A.11),

$$c^{n-1}\operatorname{Tr}(\delta S)\sum_{r=1}^{n}(-1)^{r}\binom{d-1}{r-1} = c^{n-1}\operatorname{Tr}(\delta S)(-1)^{n}\binom{d-2}{n-1}.$$
 (A.22)

Putting these together in (A.18) gives, for $n \ge 1$,

$$\delta Y_{(n)}(S)|_{\bar{S}} = (-1)^n c^{n-1} \binom{d-2}{n-1} \left[\text{Tr}(\delta S) \mathbb{1} - \delta S \right].$$
 (A.23)

From this one can easily obtain $\delta Y_{(n)}(S^{-1})|_{\bar{S}}$ by replacing $c \to 1/c$ and noting that $\delta(S^{-1}) = -c^{-2}\delta S$. Finally, putting all this together one gets,

$$\delta V_{\mu\nu}^g \big|_{\bar{S}} = \frac{m_g^{d-2}}{m^d} \Lambda_g \, \delta g_{\mu\nu} + N \, \bar{g}_{\mu\lambda} \, \left(\text{Tr}(\delta S) \delta_{\nu}^{\lambda} - \delta S_{\nu}^{\lambda} \right) \,, \tag{A.24}$$

$$\delta V_{\mu\nu}^f|_{\bar{S}} = \frac{m_f^{d-2}}{m^d} \Lambda_f c^{-2} \delta f_{\mu\lambda} - c^{2-d} N \bar{g}_{\mu\lambda} \left(\text{Tr}(\delta S) \delta_{\nu}^{\lambda} - \delta S_{\nu}^{\lambda} \right) , \tag{A.25}$$

with,

$$N = \left[\sum_{n=1}^{d-1} c^{n-1} \beta_n \binom{d-2}{n-1} \right]. \tag{A.26}$$

The Einstein tensor in (2.6) is linearized in the standard way: one has,

$$\delta\left(R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right) = \bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta g_{\rho\sigma} + \mathcal{R}_{\mu\nu}. \tag{A.27}$$

The structure of $\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta g_{\rho\sigma}$ is given by (1.2) but with the background metric $\bar{g}_{\mu\nu}$, and,

$$\mathcal{R}_{\mu\nu} = -\frac{1}{2} \left(\bar{R} \delta g_{\mu\nu} - \bar{g}_{\mu\nu} \bar{R}^{\rho\sigma} \delta g_{\rho\sigma} \right) = -\frac{\Lambda_g}{d-2} \left(d\delta g_{\mu\nu} - \bar{g}_{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{\rho\sigma} \right) \tag{A.28}$$

In the last step we have used the background equation (2.9). Finally, using (A.27) and the corresponding equation for δf , along with (A.24) and (A.25) in (A.15), we arrive at the linearized equations (2.13) and (2.14) for the fluctuations δg and δf .

B The nonlinear G- M^G action

We sketch a brief derivation of the nonlinear bimetric action (2.1) written in terms of the fields

$$G_{\mu\nu} = \left(1 + (\alpha c)^{d-2}\right)^{(4-d)/(d-2)} \left(g_{\mu\nu} + (\alpha c)^{d-2}c^{-2}f_{\mu\nu}\right), \tag{B.1}$$

and

$$M_{\mu\nu}^{G} = \left(1 + (\alpha c)^{d-2}\right)^{-2/(d-2)} \left(G_{\mu\rho}S_{\nu}^{\rho} - c G_{\mu\nu}\right), \tag{B.2}$$

where we recall that $S = \sqrt{g^{-1}f}$. These definitions are consistent with all of our considerations in the main text. From these definitions we straightforwardly obtain the inverted relations

$$g_{\mu\nu} = G_{\mu\rho}(\Phi^{-1})^{\rho}_{\ \nu}, \quad f_{\mu\nu} = G_{\mu\alpha}S^{\alpha}_{\ \lambda}S^{\lambda}_{\ \rho}(\Phi^{-1})^{\rho}_{\ \nu},$$
 (B.3)

where, for notational purposes, we have defined

$$\Phi^{\rho}_{\ \nu} = \left(1 + (\alpha c)^{d-2}\right)^{(4-d)/(d-2)} \left(\delta^{\rho}_{\nu} + (\alpha c)^{d-2} c^{-2} S^{\rho}_{\ \sigma} S^{\sigma}_{\ \nu}\right). \tag{B.4}$$

From (B.2) we can also directly obtain S in terms of $G_{\mu\nu}$ and $M_{\mu\nu}^G$,

$$S^{\rho}_{\ \nu} = c \,\delta^{\rho}_{\nu} + \left(1 + (\alpha c)^{d-2}\right)^{2/(d-2)} G^{\rho\mu} M^G_{\mu\nu} \,. \tag{B.5}$$

It is now easy to see that the volume densities can be rewritten through

$$\sqrt{g} = \sqrt{G} \det(\Phi)^{-1/2}, \quad \sqrt{f} = \sqrt{G} \det(S) \det(\Phi)^{-1/2}.$$
 (B.6)

In order to rewrite the interaction potential we note a general relation obeyed by the symmetric polynomials,

$$e_n(1+\mathbb{X}) = \sum_{k=0}^n \binom{d-k}{n-k} e_k(\mathbb{X}). \tag{B.7}$$

If we consider the matrix parameterization $S = c(1 + \mathbb{X})$ (c.f. (B.5)), we may use this relation to rewrite the interaction potential, noting that

$$V(S; \beta_n) = \sum_{n=0}^d \beta_n e_n(S) = \sum_{n=0}^d c^n \alpha_n e_n(\mathbb{X}) = V(\mathbb{X}; c^n \alpha_n),$$
 (B.8)

where the different set of coefficients are related by

$$c^{n}\alpha_{n} = \sum_{k=n}^{d} {d-n \choose k-n} c^{k}\beta_{k}, \quad c^{n}\beta_{n} = \sum_{k=n}^{d} (-1)^{n+k} {d-n \choose k-n} c^{k}\alpha_{k}.$$
 (B.9)

This is straightforward to implement in the interaction potential, which is then given by

$$V(S; \beta_n) = V(G^{-1}M^G; \tilde{\alpha}_n), \quad \tilde{\alpha}_n = \left(1 + (\alpha c)^{d-2}\right)^{2n/(d-2)} \alpha_n.$$
 (B.10)

For the kinetic terms we choose to write these in terms of covariant derivatives with respect to $G_{\mu\nu}$. Using the Riemann curvature definition $[\nabla_{\mu}, \nabla_{\nu}] \omega_{\rho} = R_{\mu\nu\rho}{}^{\sigma} \omega_{\sigma}$, together with the general relation between two covariant derivations on a manifold,

$$\nabla^g_{\mu}\omega_{\nu} = \nabla^G_{\mu}\omega_{\nu} - C_{\mu\nu}{}^{\rho}\omega_{\rho}, \quad C_{\mu\nu}{}^{\rho} = \frac{1}{2}g^{\rho\sigma}\left(2\nabla^G_{(\mu}g_{\nu)\sigma} - \nabla^G_{\sigma}f_{\mu\nu}\right), \tag{B.11}$$

we obtain the relation (with an obvious similar relation for $R_{\mu\nu}(f)$ obtained by replacing $g \to f$ in these expressions)

$$R_{\mu\nu}(g) = R_{\mu\nu}(G) - 2\nabla^G_{[\mu}C_{\rho]\nu}^{\ \rho} + 2C_{\nu[\mu}^{\ \sigma}C_{\rho]\sigma}^{\ \rho}.$$
(B.12)

Tracing this with $g^{\mu\nu}$ ($f^{\mu\nu}$) and writing it out in full we find that (modulo total derivatives and neglecting to write out an overall factor of $\sqrt{|g|}$ ($\sqrt{|f|}$) on both sides) we get Ricci curvature relations of the form

$$R(g) = g^{\mu\nu} R_{\mu\nu}(G) - \frac{1}{2} g_{\rho\lambda} \nabla_{\alpha} g^{\rho\lambda} \nabla_{\sigma} g^{\alpha\sigma} + \frac{1}{2} g_{\rho\sigma} \nabla_{\lambda} g^{\alpha\sigma} \nabla_{\alpha} g^{\rho\lambda} - \frac{1}{4} g_{\rho\kappa} g_{\lambda\beta} g^{\alpha\sigma} \nabla_{\alpha} g^{\rho\lambda} \nabla_{\sigma} g^{\kappa\beta} + \frac{1}{4} g_{\kappa\beta} g_{\rho\lambda} g^{\alpha\sigma} \nabla_{\alpha} g^{\kappa\beta} \nabla_{\sigma} g^{\rho\lambda} \equiv g^{\mu\nu} R_{\mu\nu}(G) + \Pi^g , \quad (B.13)$$

with a similar relation for R(f) and definition of Π^f , obtained by replacing $g \to f$ in this. Collecting our results we can now write the nonlinear action in terms of $G_{\mu\nu}$ and $M^G_{\mu\nu}$ as

$$S_{GM} = m_g^{d-2} \int d^d x \sqrt{|G|} \det(\Phi)^{-1/2} \Big[(\Phi G^{-1})^{\mu\nu} R_{\mu\nu}(G) + \alpha^{d-2} \det(S) (\Phi G^{-1} S^{-2})^{\mu\nu} R_{\mu\nu}(G) + \Pi^g + \alpha^{d-2} \det(S) \Pi^f - 2 \frac{m^d}{m_g^{d-2}} V(G^{-1} M^G; \tilde{\alpha}_n) \Big], \quad (B.14)$$

where the relations (B.3), (B.4), (B.5) and (B.13) can be used to get the explicit form in terms of only $G_{\mu\nu}$ and $M_{\mu\nu}^G$. Expanding this action to second order gives the quadratic action (3.25) by construction.

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